TECHNOLOGY DEPT.

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 62 NUMBER 7

PART I

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AUGUST-SEPTEMBER 1955

# The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)



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THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)



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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

# THE METHOD OF ARCHIMEDES*

S. H. GOULD, Williams College

The works of Archimedes have come down to us in two streams of tradition, one of them continuous, the other broken by a gap of a thousand years between the tenth century and the year 1906, when the discovery of a manuscript in Constantinople brought to light an important work called the *Method*, on the subject of integration.

Newton and his contemporaries in the seventeenth century were much puzzled by one aspect of the integrations to be found in the continuous tradition. In the books on the *Sphere and Cylinder*, for example, it is clear that the somewhat complicated method employed there for finding the volume of a sphere represents merely a rigorous proof of the correctness of the result and gives no indication how Archimedes was led to it originally. The discovery of 1906 removes the veil, at least to some extent.

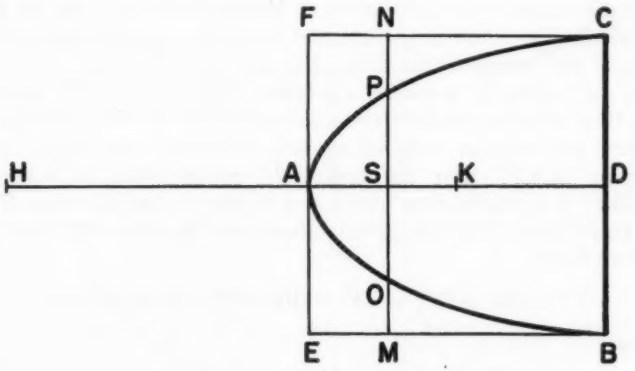


FIG. 1

The newly discovered *Method* consists of imagining the desired volume as cut up into a very large number of thin parallel slices or discs, which are then suspended at one end of an imaginary lever in such a way that they are in equilibrium with a solid whose volume and center of gravity are known. Thus, in Proposition 4 of the *Method*, Archimedes shows that the volume of a paraboloid of revolution is one-half of the volume of the circumscribing cylinder by slicing the two solids (see Figure 1 which represents a plane section through their common axis $AD$) at right angles to $AD$. For let us take $HAD$ to be the bar of a balance with $HA = AD$ and with the fulcrum at $A$, and imagine the circle $PO$ to be removed from the paraboloid and suspended at $H$. Since $AD/AS = DB^2/SO^2$ in the parabola $BAC$, we have

* An address to the Mathematical Association of America at the 1953 Summer Meeting in Kingston, Ontario, Canada.

$$\frac{HA}{AS} = \frac{AD}{AS} = \frac{MS^2}{SO^2} = \frac{\text{(circle in cylinder)}}{\text{(circle in paraboloid)}},$$

so that, by the law of the lever, the circle in the cylinder, remaining where it is, is in equilibrium with the circle from the paraboloid resting in its new position. If we deal in the same way with all the circles making up the paraboloid, we find that the cylinder, resting where it is with its center of gravity at the midpoint $K$ of $AD$, is in equilibrium about $A$ with the paraboloid placed with its center of gravity at $H$. Since $HA = AD = 2AK$, the volume of the paraboloid is therefore one-half of that of the cylinder, as desired.

Many accounts of the *Method* have been given since its discovery in 1906; for example, by T. L. Heath in his *Supplement to the Works of Archimedes*, Cambridge, 1912. In all of them, as in the original work of Archimedes himself, we are invited to *imagine* the lever and the objects suspended from it. But if we construct an *actual* lever and *actual* discs, the various figures, which may be spheres, cones, *etc.*, see below, will be observed to balance, slice by slice, as successive slices are added. The whole procedure then becomes a picturesque and effective illustration of the concept of an integral as the limit of a sum.

To find the volume of a sphere, a problem which Archimedes considered so important that he asked to have the result engraved on his tombstone, a cone and a sphere are together weighed against a cylinder (see Figure 2 and the accompanying sketch). Here the circle $NM$, resting where it is in the large cylinder $GLEF$ is in equilibrium about $A$ with two circles placed at $H$, the one circle $PO$ being taken from the given sphere and the other $RQ$ from the cone $FAE$. For we have

$$OS^2 + QS^2 = OS^2 + AS^2 = AO^2 = CA \cdot AS = MS \cdot SQ$$

and therefore

$$\frac{HA}{AS} = \frac{MS}{SQ} = \frac{MS^2}{MS \cdot SQ} = \frac{MS^2}{OS^2 + QS^2}.$$

Thus, by the law of the lever as before,

*one-half of cylinder equals cone plus sphere*

from which, since the cone is one-third of the cylinder,

*sphere equals one-sixth cylinder.*

Thus the cylinder circumscribed about the sphere, being one-quarter as great as the large cylinder $GLEF$, is three-halves as great as the sphere, which is the result stated on the tombstone of Archimedes.

If squares are substituted for the circles of cross-section in these figures, the argument remains unchanged and we have the solution of another famous problem (Proposition 15 in the *Method*), namely to find the volume common to

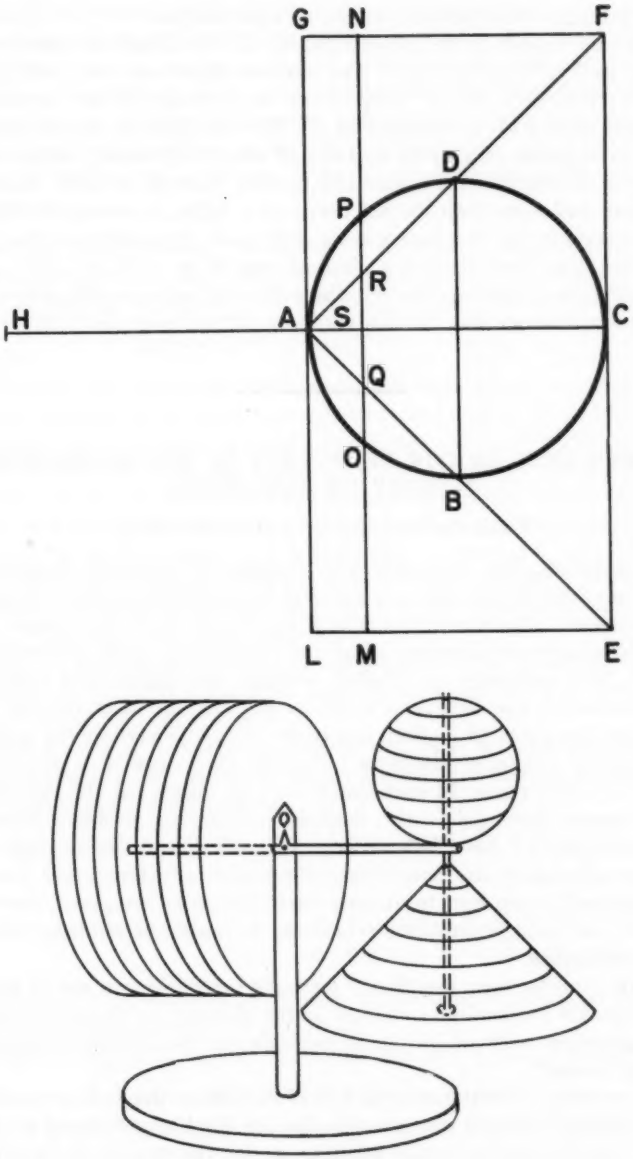


FIG. 2

two right circular cylinders intersecting at right angles.

The actual models were constructed by D. A. Eberle of the Psychology Workshop at Purdue University. The various slices were cut from a piece of white pine 1/2″ thick and 7″ wide. Thus the cylinder *GLEF* is composed of seven slices, each with a diameter of 7″. The seven slices for the cone, being first cut as stepwise increasing cylindrical discs with easily calculated radii, were placed all together on a mandrel passing through a 3/16″ hole through their centers and were then shaped down on a lathe, a procedure found to be especially necessary for the square cross-sections in the problem of the intersecting cylinders. The lever itself is a piece of steel 9″ by 1/2″ by 1/32″, placed so that its 1/2″ face is vertical. In each disc a thin slit was cut with a fine hacksaw from edge to center so that the disc could be slipped onto the lever.

## SIMPLE DEVICES FOR EFFICIENCY IN THE ELEMENTARY THEORY OF EQUATIONS

FLORA DINKINES, University of Illinois, Chicago

**1. Introduction.** The instructor who attempts to teach his students to find the "best" method for solving each problem frequently finds that the process of producing several solutions, from which the "best" is to be chosen, tends to stimulate ingenuity and produce a habit of efficiency in most students. Not infrequently, it is necessary to consider whether the author has indicated the "best" procedure in the directions given, or whether the ultimate aim—such as finding all of the roots of a given equation—can be more readily attained by disregarding the author's directions involving some specific method—such as testing for rational roots. In such cases shall we disregard the directions and obtain the results more efficiently; shall we consider the problem satisfactorily worked only when we have followed the author's directions; or shall we work the problem efficiently and then follow the author's instructions? The answer probably depends upon how frequently such directions are given, and whether or not there are enough problems remaining to teach the students the desired standard techniques.

We shall consider here the "best" methods for working some of the college algebra problems ordinarily contained in the chapter on theory of equations, and the frequency with which these methods can be applied to problems in certain text books.*

Before entering upon the main points of discussion the author would like to say that no special method was used in selecting the books referred to. Those at hand were merely used to decide whether or not the devices to be given here

* These books are listed at the end of this article and referred to by number.

could be applied frequently enough to make their consideration worthwhile.

It is obvious that a special device, even though it is a great time saver, should be very easy to teach, or should be frequently applicable, in order to merit its being taught in preference to, or in addition to, the standard or more general procedures. In view of this the author believes that two devices here presented should be considered as standard techniques in any treatment, however brief, of the chapter on theory of equations presented in the usual college algebra course.

**2. The two main devices.** Throughout this paper we shall confine ourselves to polynomial equations with real coefficients. Usually the coefficients will be integers. Such a polynomial we shall represent in the usual manner by

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.$$

The first device arises from the observation that if $\sqrt{a}$ is a root of such a polynomial equation with integral coefficients it must be a root of the polynomial consisting of the even powered terms only, and also a root of the polynomial consisting of the odd powered terms only. This follows from the fact that the substitution of $\sqrt{a}$ into a term produces a multiple of $\sqrt{a}$ if the term is of odd power and an integer if it is of even power. This suggests grouping the odd powers and the even powers and looking for common factors. This simple method frequently makes the equation at hand appear rather transparent, and it is interesting to find that it frequently reveals factors other than those of the type $x^2-a$, where $a$ is not a perfect square.

The following examples indicate the usefulness of this type of factoring. (We shall consider the problem completed when we have reduced it to the solution of one or more quadratic equations.)

*Example* 1. [7], p. 221, no. 13. The instructions are to solve the equation given that $i$ and $-1+3i$ are roots. We ignore the instructions and regroup the equation into odd and even powers.

$$\begin{aligned} x^6 + 2x^5 + 10x^4 - x^2 - 2x - 10 &= x^6 + 10x^4 - x^2 - 10 + 2x(x^4 - 1) \\ &= (x^2 + 10)(x^4 - 1) + 2x(x^4 - 1) \\ &= (x^4 - 1)(x^2 + 2x + 10) = 0, \\ x &= \pm 1, \quad \pm i, \quad -1 \pm 3i. \end{aligned}$$

Note that this equation can be completely solved without using the information given or any information on rational roots.

*Example* 2. [7], p. 200, no. 17. The instructions are to find all rational roots.

$$9x^4 - 30x^3 + 16x^2 + 30x - 25 = 0.$$

$$\begin{aligned} 9x^4 + 16x^2 - 25 - 30x^3 + 30x &= (9x^2 + 25)(x^2 - 1) - 30x(x^2 - 1) \\ &= (9x^2 - 30x + 25)(x^2 - 1) = 0. \\ x &= 5/3, \quad 5/3, \quad \pm 1. \end{aligned}$$

All roots can be found without using the test for rational roots.

*Example* 3. [2], p. 376, no. 13. The instructions are to find all roots. As usual, it is assumed that the student must use the test for rational roots to reduce this to a quadratic.

$$\begin{aligned} 15y^4 + 4y^3 + 56y^2 + 16y - 16 &= 15y^4 + 56y^2 - 16 + 4y^3 + 16y \\ &= (y^2 + 4)(15y^2 - 4) + 4y(y^2 + 4) \\ &= (15y^2 + 4y - 4)(y^2 + 4) = 0. \end{aligned}$$

*Examples* 4–7. [6], p. 247, nos. 1, 4, 6, 7. Here the instructions are to approximate a specific root to the nearest hundredth. Each of these is readily factored into the product of two quadratics, by this first device, and *all* roots can readily be obtained.

*Example* 8. [5], p. 262, no. 8. The instructions are to find the rational roots. There are none. This method of factoring quickly reveals that there are none, and also produces *all* of the roots. No synthetic division is necessary.

$$x^5 + 5x^3 - 7x^2 - 35 = 0.$$

$$\begin{aligned} x^3(x^2 + 5) - 7(x^2 + 5) &= (x^3 - 7)(x^2 + 5) \\ &= (x - \sqrt[3]{7})(x^2 + \sqrt[3]{7}x + \sqrt[3]{49})(x^2 + 5) \end{aligned}$$

*Example* 9. [5], p. 269, no. 8. The instructions are to compute, correct to three decimal places, the larger of the two irrational roots of this equation. $x^4-4x^3+5x^2-16x+4=0$. But why compute when we can factor the equation into two quadratics and thus obtain the exact answers?

$$\begin{aligned} x^4 + 5x^2 + 4 - 4x^3 - 16x &= (x^2 + 4)(x^2 + 1) - 4x(x^2 + 4) \\ &= (x^2 + 4)(x^2 - 4x + 1) = 0. \end{aligned}$$

Also, why compute for [5], p. 276, nos. 21, 22?

*Example* 10. [1], p. 324, no. 18. Here the instructions are to find all rational roots. If the depressed equation is a quadratic, find all roots.

$$x^5 - 10x^4 + 32x^3 - 38x^2 + 31x - 28 = 0.$$

One readily finds that 4 is the only rational root. This is the answer given by the author. However, since the depressed equation is $x^4-6x^3+8x^2-6x+7=0$, which factors by this first device into $(x^2+1)(x^2-6x+7)=0$, *all* roots are readily obtained.

*Example* 11. [1], p. 336, Illustrative Example. Here the author explains how to apply Ferrari's solution of the quartic to the equation

$$x^4 - 4x^3 + 6x^2 - 4x + 5 = 0,$$

which factors by this first device into $(x^2+1)(x^2-4x+5)=0$. Example 10 on this page also factors readily. However, it should be admitted, in all fairness to the author, that it is frequently desirable to teach the solution of the quartic by

means of a simple example, but is it not imperative that we not mislead the student into thinking that there is no simpler method of working this specific problem?

*Examples* 12–14. [4], p. 242, Examples 1 and 2; p. 244, Example 1.

Here the author has chosen equations which could be factored by this first device to use in explaining methods for finding rational roots, or all of the roots.

$$x^4 - 6x^3 + 3x^2 + 24x - 28 = (x^2 - 4)(x^2 - 6x + 7).$$

$$3x^3 + 2x^2 - 3x - 2 = (3x + 2)(x^2 - 1).$$

$$64x^3 - 16x^2 + 12x - 3 = (4x - 1)(16x^2 + 3).$$

*Example* 15. As a last example we point out that an equation like

$$x^6 - x^5 + 4x^4 - 2x^3 - 31x^2 + 35x - 70 = 0,$$

which has no rational roots, is readily solved by this method. Regrouping and factoring the odd powered terms, we have $x^6+4x^4-31x^2-70-x(x^4+2x^2-35)$. In factoring the even powered terms we merely test to see if $x^4+2x^2-35$ is a factor. No division is necessary since we observe that no odd powers can be obtained in the multiplication of the factors; therefore the second factor must be a binomial, which is obviously $x^2+2$. After checking the multiplication we have

$$(x^2 + 2)(x^4 + 2x^2 - 35) - x(x^2 + 2x^2 - 35)$$
$$= (x^2 - x + 2)(x^4 + 2x^2 - 35) = (x^2 - x + 2)(x^2 + 7)(x^2 - 5) = 0.$$

$$x = \pm i\sqrt{7}, \qquad x = \pm \sqrt{5}, \qquad x = \frac{-1 \pm i\sqrt{7}}{2}.$$

With a little effort the author was able to find the following problems in the listed text books for which this method will give a complete solution or will aid in the solution. Although these books have not been thoroughly checked, this should indicate that there are many problems for which we are teaching unnecessarily difficult methods of solution, or stated differently, there are many for which we are not teaching the most efficient methods of solution.

*Problems to which the first device can be applied.*

[1], p. 324, nos. 16, 18; p. 336, Example 1, nos. 7, 10.

[2], p. 376, nos. 13, 15.

[3], p. 46, no. 2; p. 51, nos 1, 2; p. 52, nos. 1, 4.

[4], p. 243, no. 7; p. 242, Examples 1, 2; p. 244, Example 1, no. 15; p. 248, no. 4; p. 257, nos. 8, 9, 10.

[5], p. 246, no. 20; p. 252, no. 39; p. 254, Example 2; p. 258, nos. 25, 26, 28; p. 262, nos. 1, 6, 8, 12, 13, 15, 16, 17, 19, 20, 21, 22; p. 269, no. 8; p. 276, nos. 21, 22.

[6], p. 221, nos. 31, 33, 34; p. 223, nos. 5, 9, 10; p. 226, no. 29; p. 229, nos. 11, 12, 13, 14, 15, 27; p. 234, nos. 4, 7, 11, 13, 15, 19, 20, 21, 22, 24, 27, 33, 34; p. 239, nos. 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16; p. 244, nos.

27, 28; p. 247, nos. 1, 4, 6, 7; p. 255, nos. 17, 18, 19, 20; p. 260, no. 2.

[7], p. 200, no. 17; p. 221, no. 13.

[8], p. 63, no. 1(g).

The second device is merely a way of factoring a quartic equation with integral coefficients into the product of two quadratic factors with integral coefficients. Assume that $ax^4+bx^3+cx^2+dx+e=0$ can be factored into

$$(mx^2 + kx + r)(nx^2 + tx + s) = 0.$$

Then

$$ax^4 + bx^3 + cx^2 + dx + e = mnx^4 + (mt + nk)x^3 + (tk + nr + ms)x^2 + (rt + sk)x + rs.$$

If now, we choose two integers $m$ and $n$ whose product is $a$, and two integers $r$ and $s$ whose product is $e$, we can use these values in the linear equations obtained by equating the coefficients of $x^3$ and $x$. Thus we seek integers $t$ and $k$ satisfying

$$\text{(1)} \qquad \begin{cases} mt + nk = b \\ rt + sk = d. \end{cases}$$

If integral values can be found which also check in the equation

$$\text{(2)} \qquad tk + nr + ms = c,$$

obtained by equating the coefficients of the second degree terms, then the quartic has the factors $mx^2+kx+r$ and $nx^2+tx+s$, as desired.

This method is especially useful when $a=1$, for then the equations reduce to

$$\text{(3)} \qquad \begin{cases} t + k = b \\ rt + sk = d \end{cases}$$

and

$$\text{(4)} \qquad tk + r + s = c,$$

and one can try all possible integral factorizations of $e$.

Some examples of the use of this second device are given here.

*Examples* 1–6. [1], p. 331, nos. 7, 8, 9, 10, 16, 18.

For nos. 7 and 8 the instructions are to find specified roots correct to three decimal places. For no. 7, $x^4+2x^3-8x^2-6x-1=0$, the roots indicated are between 2 and 3, and between $-3$ and $-4$.

We ignore the author's directions and choose $r=-1$, $s=1$. Then

$$\begin{cases} t + k = 2 \\ -t + k = -6, \end{cases}$$

and we obtain $k=-2$, $t=4$. These values check $tk+r+s=-8$, and the quadratic factors are $x^2-2x-1$ and $x^2+4x+1$. From these we can get the answers in

exact form, or their decimal approximations, if these are needed.

For no. 8, $x^4-6x^2-12x-8=0$, the roots indicated are between 3 and 4, and between $-1$ and $-2$. Since $b=0$, (3) becomes

$$\begin{cases} t+\ k=0 \\ rt+sk=-\ 12, \end{cases}$$

from which we obtain $t=-k$ and $k=12/(r-s)$. From (4) we have that $k^2=r+s+6$, for which it is easy to see that $r=-4$, and $s=2$ are integral solutions. Then the sign of $k$ is given by $k=12/(r-s)$, and the factors are $x^2-2x-4$ and $x^2+2x+2$. (The same type of solution readily produces the factors of [6], p. 239, no. 10: $x^4+17x^2-30x-144=0$).

The instructions for nos. 16 and 18 are to find all real roots exactly or correctly to two decimal places.

No. 16, $2x^4-5x^3-33x^2-4x+40=0$, can be broken into the factors $2x^2+3x-5$ and $x^2-4x-8$, by this second device, and the four real roots can be obtained without using any method of approximation.

No. 18, $2x^5+x^4-16x^3+27x^2-14x-6=0$, cannot be easily factored. We use the usual procedure for rational roots to locate the root 3/2. The depressed equation $x^4+2x^3-5x^2+6x+2=0$ readily factors into $x^2-2x+2$ and $x^2+4x+1$ by the second device. Thus again the longer methods of approximation have been avoided.

*Example* 7. [5], p. 276, no. 26 can be readily factored by this method. We shall not repeat the author's lengthy instructions.

$$x^4 - 9x^3 + 14x^2 + 13x + 23 = (x^2 - 10x + 23)(x^2 + x + 1) = 0.$$

*Examples* 8–17. [7], p. 229, nos. 1–10. Here the instructions are to solve the quartic equations by the method outlined in the test. This is Ferrari's solution of the quartic, though the author doesn't say that. Every one of these problems can be solved by this second device and some of them very easily, such as number 2: $x^4+6x^3-8x^2-22x-105=(x^2+4x-21)(x^2+2x+5)=0$.

*Example* 18. [5], p. 281, no. 18. By means of this device the resulting equation $27x^4-45x^3-15x^2+24x+16=0$ can be factored into

$$(9x^2 - 24x + 16)(3x^2 + 3x + 1) = 0.$$

The frequency with which the second device can be applied is indicated by the following set of problems. Again the author wishes to indicate that the supply has not been exhausted.

*Problems to which the second device can be applied.*

[1], p. 321, nos. 13, 14, 16; p. 323, nos. 3, 4, 5, 6, 11, 14; p. 331, nos. 7, 8, 9, 10, 15, 16, 18.

[2], p. 376, nos. 8, 9; p. 378, Example 1.

[3], p. 46A (quartics), nos. 1, 2, 3, 4, 5, 6; p. 46B, no. 55; p. 51, nos. 3, 4, 5.

[4], p. 243, no. 8; p. 257, nos. 6, 7.

[5], p. 246, no. 23; p. 251, nos. 32, 35; p. 258, no. 27; p. 263, no. 23; p. 275, no. 5; p. 276, nos. 23, 24, 25, 26; p. 281, nos. 4, 9, 10, 11, 12, 13, 14, 15, 18.

[6], p. 223, nos. 6, 7, 8; p. 226, nos. 30, 31; p. 230, no. 28; p. 234, nos. 5, 6, 9, 14, 31, 32, 35, 36; p. 239, no. 10; p. 255, nos. 25, 26; p. 262, no. 4.

[7], p. 200, nos. 12, 15, 26; p. 204, nos. 6, 15; p. 213, no. 20; p. 229, nos. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

[8], p. 63, nos. 1(c), (d).

**3. Miscellaneous devices.** The following devices are helpful though they are not as frequently applicable as the two main devices.

(a) Factoring by grouping terms other than even and odd powered terms.
[3], p. 46, no. 2. $x^3+6x^2+3x+18=x^2(x+6)+3(x+6)=(x^2+3)(x+6)$.
[5], p. 262, nos. 3, 7; p. 246, no. 18; p. 286, nos. 4, 7.
[6], p. 221, no. 32; p. 234, no. 8.
[7], p. 200, no. 3.
[8], p. 63, no. 1(a).

(b) Completing the square on $ax^4+bx^2$.
[1], p. 331, no. 9; p. 336, no. 7.
[6], p. 255, no. 25. (We explain this problem since it saves the reader the task of finding 4 real roots by Horner's method.)

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$$
$$x^4 - 6x^3 + 9x^2 = 9x^2 - 8x^2 - 2x + 1$$
$$(x^2 - 3x)^2 = (x - 1)^2, \qquad x^2 - 3x \pm (x - 1) = 0,$$
$$x^2 - 2x - 1 = 0, \qquad x^2 - 4x + 1 = 0.$$

(c) Factoring as the difference between two squares.
[1], p. 336, no. 8.
[3], p. 46B, no. 32; $x^4=\pm(4x+1)^2$.
[5], p. 258, no. 29; $x^2=\pm(2x-3)$. p. 263, no. 27; $(x^2-7)^2=57$. p. 275, no. 6; $(x^2+4x)^2=25$.
[8], p. 93, no. 1(d). $x^2=\pm\sqrt{2}(x-3)$.

(d) Factoring of the type $(ax^2+bx)^2+c(ax^2+bx)+d$.
[1], p. 324, no. 14.

$$x^6 - 8x^5 + 25x^4 - 36x^3 + 20x^2 = 0$$
$$x^2(x^4 - 8x^3 + 16x^2 + 9x^2 - 36x + 20) = 0$$
$$x^2[(x^2 - 4x)^2 + 9(x^2 - 4x) + 20] = 0$$
$$x^2(x^2 - 4x + 5)(x^2 - 4x + 4) = 0.$$

(e) Factoring involving the sum or difference of two cubes.
[3], p. 46, no. 3;

$$y^3 - 2y + 4 = y^3 + 8 - 2y - 4 = (y + 2)(y^2 - 2y + 4) - 2(y + 2)$$
$$= (y + 2)(y^2 - 2y + 2).$$

[6], p. 223, no. 3;

$$x^3 + 4x^2 + 8x + 8 = x^3 + 8 + 4x^2 + 8x = (x + 2)(x^2 - 2x + 4) + 4x(x + 2)$$
$$= (x + 2)(x^2 + 2x + 4).$$

**4. Suggestion to authors of college algebra texts.** The author of this article suggests that the authors of college algebra books which are to be used as text books devote a paragraph near the beginning of the chapter on theory of equations to devices such as are discussed here. Then if the following exercises contain problems to which special devices apply the student can save time by applying them. Needless to say there are many problems which demand some method of approximating roots, or the method for finding rational roots. Is it not sound teaching to strive for efficiency?

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*Editorial Note.* This paper illustrates the artificiality of most of our instruction in the Theory of Equations. The solution of a polynomial equation with arbitrary real coefficients is usually a messy business which can be handled only by some method of approximation. Our textbook examples, however, are selected to have relatively "nice" answers. This gives the students a false impression of the true difficulties of the problem.

Because of the complications of an approximate solution, it is certainly desirable to look for linear or quadratic factors before using a method of approximation. The theorem on rational roots is designed to identify certain linear factors, and the devices mentioned by Professor Dinkines are helpful in finding certain quadratic factors. The student should be warned, however, that these methods are much more likely to be of use in textbook problems than in problems derived from scientific situations. The probability that a polynomial with integral coefficients will factor into polynomials of lower degree with integral coefficients is zero!

I believe that the authors of the textbooks mentioned above intended to

present exercises which illustrate the need for approximate methods. In constructing their equations, however, they seem to have worked backwards and to have formed their polynomials by multiplying together linear and quadratic factors. This made it easy for them to know the exact answers, which the students were expected to approximate by other methods. As a further hint to authors of textbooks, let me suggest that exercises intended for approximate solution be so constructed that the more elementary methods of solution mentioned above are not applicable.

C.B.A.

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## CORRECTION

LEE LORCH, Fisk University

In the article *The principal term in the asymptotic expansion of the Lebesgue constants*, this MONTHLY, vol. 61, 1954, pp. 245–249, the following correction should be made: on page 246, line 29, sin $t$ should read $|\sin t|$.

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# MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

*Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.*

## A NOTE ON THE RECURRING PERIOD OF THE RECIPROCAL OF AN ODD NUMBER

K. SUBBA RAO, Maharajah's College, Vizianagram

In this paper, I show how to obtain the recurring period of $1/n$, where $n$ is any odd number and $(n, 5)=1$, by multiplication instead of by division. I also derive the form of $n$ in order that $1/n$ may have the maximum recurring period $\phi(n)$, where $\phi(n)$ is Euler's totient function. I further append a few results to Daljit Singh's results [1] concerning the reciprocal of a prime.

**1.** It has been shown by D. R. Kaprekar [2] that if $p$ be any odd prime, other than 5, and $a$ be the least integer such that $N(=ap+1)$ is divisible by 10, then the last $(p-1)$ digits in

$$\frac{N^{r+1}-1}{p} = a(1+N+N^2+\cdots+N^r), \qquad r \geqq p-1,$$

will represent a recurring period of the fraction $1/p$. These $(p-1)$ digits will

either be the actual recurring period of $1/p$ or will contain the recurring portion an integral number of times. $a$ has the values 9, 3, 7 or 1 according as the prime ends in 1, 3, 7, or 9. The number $a$ is called the first figure and the number $N/10$ is called the constant multiplier, and Kaprekar has shown how the recurring period of $1/p$ can be obtained by the "carrying forward" process. The same process can be applied to obtain the recurring period of $1/n$, where $n$ is any odd number, not necessarily a prime, and $(n, 5)=1$. For, 10 is prime to $n$ and if 10 belongs to the exponent $e$ modulo $n$, $10^e-1$ is divisible by $n$, that is, a number with $e$ nines will be divisible by $n$ and the number of digits in the recurring period of $1/n$ is $e$. Thus if $n$ is of the form $(10m+1)$, $a=9$ and $N=90m+10$; the first number is 9, the constant multiplier is $(9m+1)$ and the "carrying forward" process gives the recurring period of $1/n$. Similarly when $n$ is of the forms $(10m+3)$, $(10m+7)$ and $(10m+9)$, $a$ takes the values 3, 7 and 1 respectively, and the constant multiplier has the values $(3m+1)$, $(7m+5)$ and $(m+1)$ respectively.

**2.** A. A. Krishnaswami Ayyangar [3] has given a set of conditions which must hold in order that $q$ may be a prime with a maximum recurring period $(q-1)$ for $1/q$. I now discuss the corresponding problem when $q$ is an odd number. I prove the following

THEOREM. *If $n$ is an odd number with a maximum recurring period $\phi(n)$ for $1/n$, then $n$ must be of the form $p^m$, where $m \geqq 1$ and $p$ is a prime other than* 5.

*Proof.* From the foregoing discussion, we see that if $e$ is the exponent to which 10 belongs modulo $n$, then the recurring period consists of $e$ digits. If $e=\phi(n)$, then the recurring period is the maximum and it is equal to $(n-1)$, if $n$ is an odd prime $(\neq 5)$. It follows that the recurring period of $1/n$ is a maximum when 10 is a primitive root of $n$. But it is known [4] that there exist primitive roots of a number only when it is 2, 4, $2p^m$ or $p^m$, where $p$ is an odd prime. Since $n$ is odd, it follows that $n$ must be of the form $p^m$ and the theorem is proved.

COROLLARY. *Every number of the form $p^m$ ($p$ an odd prime, other than* 5, *and $m \geqq 1$) which has* 10 *for a primitive root has the maximum recurring period.*

**3.** THEOREM. *There are infinitely many odd numbers whose reciprocals have the maximum recurring period.*

*Proof.* By the theorem in paragraph 2 above, every number of the form $p^m$ ($p$ an odd prime, other than 5, and $m \geqq 1$) which has 10 for a primitive root, has the maximum recurring period. It is known that if $g$ is a primitive root of the odd prime $p$, and if the number $g^{p-1}-1$ is not divisible by $p^2$, then $g$ is a primitive root of $p^m$, for any positive exponent $m$. If we now consider the least odd prime 7 which has 10 for a primitive root, it can easily be seen that $10^6-1$ is not divisible by 49 $(=7^2)$, from which it follows that 10 is a primitive root of $7^m$ for any positive exponent $m$. Hence the theorem is proved.

**4.** It has been shown [5] that the recurring period of 1/109 can be derived

from the first 108 terms of the Fibonacci series: 1, 1, 2, 3, 5, 8, $\cdots$ by multiplying them in order by 1, 10, $10^2$, $\cdots$ and taking their sum up to 108 digits reckoning from the unit's digit. Since the number 109 is of the form $10m+9$, we have, from paragraph 1 above, $a=1$ and the constant multiplier is 11. By putting 1 as the first figure, multiplying it by 11 and applying the "carrying forward" process, we get all the 108 digits in the recurring period of 1/109. Kaprekar states [5] that "the recurring period of 1/109 can thus be built up from the terms of the Fibonacci series and it is worthwhile investigating the result of applying a similar method to the series of Lucas: 1, 3, 4, 7, 11, 18, 29, $\cdots$." It may be noted that in the recurring period of 1/109, the digits from the 101st onwards (reckoned from right to left) are the numbers of Lucas. In fact, the series of Lucas is also of the Fibonacci type and all types of Fibonacci series can be found in the recurring period of 1/109, if it is written twice. Thus we do not get anything new by applying the "carrying forward" process to Lucas numbers. It is highly remarkable that by applying this process to all types of Fibonacci series we are led to the recurring period of 1/109 and it is to be noted that 109 is the only prime having this property.

**5.** An appendix to Daljit Singh's results *Concerning the reciprocal of a prime* [1]: by the Theorem in paragraph 2 of this paper, it follows that for all numbers of the form $p^m$ ($p$ an odd prime $\neq 5$, $m \geqq 1$) with a maximum recurring period, Singh's results hold true. I give below an example to illustrate the properties stated in Singh's paper for numbers of the form $10n+9$ ($=p^m$).

$p^m = 7^2 = 49.$

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q(a)$ | 2 | 0 | 4 | 0 | 8 | 1 | 6 | 3 | 2 | 6 | 5 | 3 | 0 | 6 |
| $r(a)$ | 2 | 20 | 4 | 40 | 8 | 31 | 16 | 13 | 32 | 26 | 15 | 3 | 30 | 6 |
| $a$ | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| $q(a)$ | 1 | 2 | 2 | 4 | 4 | 8 | 9 | 7 | 9 | 5 | 9 | 1 | 8 | 3 |
| $r(a)$ | 11 | 12 | 22 | 24 | 44 | 48 | 39 | 47 | 29 | 45 | 9 | 41 | 18 | 33 |
| $a$ | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 |
| $q(a)$ | 6 | 7 | 3 | 4 | 6 | 9 | 3 | 8 | 7 | 7 | 5 | 5 | 1 | 0 |
| $r(a)$ | 36 | 17 | 23 | 34 | 46 | 19 | 43 | 38 | 37 | 27 | 25 | 5 | 1 | 10 |

In this particular case, there exist exactly 8 pairs $q(a)=r(a)=b$ where $b$ takes all values from 1 to 9 except the value 7. This should be the case since 7 is not prime to $7^2$. For all other values of $p(\neq 5)$ and if $p^m$ is of the form $10n+9$ and has 10 for a primitive root, there exist exactly 9 pairs $q(a)=r(a)=b$ where $1 \leqq b \leqq 9$. By the Theorem in paragraph 3 above, it follows that there are infinitely many numbers of the form $(10n+9)$ having the property stated above.

6. Very little is known about those primes whose reciprocals have the maximum recurring period. Hardy and Wright in their *Introduction to the Theory of Numbers*, page 114, give the first six such primes. D. R. Kaprekar and A. A. Krishnaswami Ayyangar point out two more such primes [5], namely 109 and 487. Since all primes having 10 for a primitive root possess (by the foregoing discussion) this property, it follows from the Table of page 300 of Trygve Nagell's *Introduction to Number Theory* that, among the first 150 odd primes, there are 53 primes whose reciprocals have the maximum recurring period. It seems very likely that there are infinitely many odd primes having this property, though I am unable to prove it.

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## CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

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### NEWTON-COTES QUADRATURE FORMULAS

D. S. GREENSTEIN, University of Pennsylvania

In a recent note in this MONTHLY,* Morduchow pointed out the absence in the literature of a simple proof that the Newton-Cotes quadrature formula for $2n+1$ points gives exact results for polynomials of degree not exceeding $2n+1$. He also gave a proof which depends on Newton's forward interpolation formula.

The present note presents a simple proof depending only on the fundamental theorem of algebra. This proof might be presented to an elementary calculus class at the time Simpson's rule is discussed.

Let $f(x)$ be a polynomial of degree not exceeding $2n+1$. Let $p(x)$ be that polynomial of degree not exceeding $2n$ which interpolates $f(x)$ at the points

* Morduchow, Morris, A note on Newton-Cotes quadrature formulas, this MONTHLY, vol. 62, 1955, p. 33.

$x_0, x_0+h, x_0+2h, \cdots, x_0+2nh$ (*i.e.*, $f(x_0+ih)=p(x_0+ih)(i=0, \cdots, 2n)$). We wish to show that

$$\int_{x_0}^{x_0+2nh} f(x)dx = \int_{x_0}^{x_0+2nh} p(x)dx.$$

We shall prove this by showing that the integral between the same limits of the error polynomial $Q(x)=f(x)-p(x)$ is zero.

Since $Q(x)$ is at most of degree $2n+1$ and $Q(x_0+ih)=0$ $(i=0, \cdots, 2n)$, it follows that

$$Q(x) = C(x - x_0)(x - x_0 - h) \cdots (x - x_0 - 2nh).$$

Making the substitution $u=x-x_0-nh$, we find that

$$\int_{x_0}^{x_0+2nh} Q(x)dx = C\int_{-nh}^{nh} u(u^2 - h^2)(u^2 - 4h^2) \cdots (u^2 - n^2h^2)du,$$

which clearly vanishes, since we are integrating an odd function between symmetric limits.

## ON THE CAUCHY CONVERGENCE CRITERION

BURNETT MEYER, University of Arizona

The following proof of the Cauchy convergence criterion seems somewhat simpler than those commonly in use in that Dedekind's Theorem is used directly. No use is made of the upper and lower limits or of the Bolzano-Weierstrass Theorem. It is assumed that a Cauchy sequence and the limit of a sequence have been defined.

THEOREM: *If* $\{x_n\}$ *is a Cauchy sequence of real numbers, then there is a real number* $x$ *such that* $\lim_{n\to\infty} x_n=x$.

*Proof.* For every $\delta>0$, there is an $N$ such that if $m, n>N$, $|x_m-x_n|<\delta/2$. Hence, for all $m>N$, $x_{N+1}-\delta/2<x_m<x_{N+1}+\delta/2$.

Consider the following partition of the real numbers: In class $L$ place all real numbers $y$ such that $y$ is less than an infinite number of the $x_n$; in $R$ are all $y$ such that $y$ is less than at most a finite number of the $x_n$. Since $(x_{N+1}-\delta/2)\in L$ and $(x_{N+1}+\delta/2)\in R$, neither $L$ nor $R$ is empty. Hence, the above partition is a cut which determines a real number $x$, the largest element of $L$ or the smallest of $R$.

Since $(x-\delta/2)\in L$ and $(x+\delta/)2\in R$, there are an infinite number of the $x_n$ in the open interval $(x-\delta/2, x+\delta/2)$. Let $x_m$ be an element of the sequence in this interval with $m>N$, and let $x_n$ be any element of the sequence with $n>N$. Then

$$|x - x_n| \leqq |x - x_m| + |x_n - x_m| < \delta/2 + \delta/2 = \delta,$$

so that $\lim_{n\to\infty} x_n=x$.

## A SEQUENCE DEFINED BY A DIFFERENCE EQUATION

LOUIS BRAND, University of Cincinnati

The sequences defined by a difference equation of Riccati type have the most diverse and interesting properties, especially as regards the distribution of their cluster points. For this reason and in view of their relation to continued fractions, it is hoped that the following elementary treatment will be of service in supplying examples of how sequences behave.

The difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \qquad c \neq 0, \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, \tag{1}$$

may be reduced to the form

$$y_{n+1} = \alpha - \frac{\beta}{y_n}, \qquad \alpha = \frac{a+d}{c}, \qquad \beta = \frac{D}{c^2}, \tag{2}$$

by the substitution $x_n = y_n - d/c$. When $y_1$ is given, we shall examine the character of the sequence defined by (2). When $y_n = 0$, $y_{n+1}$ is not defined; in this case we write $y_{n+1} = \infty$, $y_{n+2} = \alpha$, and shall call the sequence *convergent* if its terms from then on have but a single cluster point.

Evidently $y_{n+1}$ can be written as a continued fraction with $n$ $\beta$'s:

$$y_{n+1} = \alpha - \frac{\beta}{\alpha -}\,\frac{\beta}{\alpha -} \cdots \frac{\beta}{y_1}. \tag{3}$$

But instead of dealing with this continued fraction we prefer to reduce (2) to a linear difference equation. Putting $y_n = z_{n+1}/z_n$ in (2) we have

$$z_{n+2} - \alpha z_{n+1} + \beta z_n = 0. \tag{4}$$

If $k_1$, $k_2$ are the roots of its characteristic quadratic

$$k^2 - \alpha k + \beta = 0, \tag{5}$$

namely $\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$, the general solution of (4) is

$$z_n = C_1 k_1^n + C_2 k_2^n \qquad \text{when } k_1 \neq k_2, \tag{6}$$

$$z_n = (C_1 + C_2 n) k_1^n \qquad \text{when } k_1 = k_2. \tag{7}$$

Since $y_n = z_{n+1}/z_n$, $y_n$ depends only on *one* essential constant, which is fixed by the value of $y_1$.

We shall confine our discussion of (2) to the case when $\alpha$, $\beta$ and $y_1$ are real; then $\{y_n\}$ is a real sequence. There are three cases to consider depending on the character of the roots $k_1$, $k_2$.

*Case* 1. $\alpha^2>4\beta$, $k_1$, $k_2$ *real and unequal.*

The general solution of (2) is, from (6),

$$y_n = \frac{C_1k_1^{n+1} + C_2k_2^{n+1}}{C_1k_1^n + C_2k_2^n}. \tag{8}$$

When $C_1=0$, $y_n=k_2$ for all $n$; and if $C_2=0$, $y_n=k_1$ for all $n$. When $y_1\neq k_1$ or $k_2$, neither constant is zero; and

$$y_n = k_2\frac{(k_1/k_2)^{n+1} + C}{(k_1/k_2)^n + C}, \qquad C \neq 0. \tag{9}$$

If $\alpha=0$ the roots of (5) are $\pm\sqrt{-\beta}$, $k_1/k_2=-1$ and (9) shows that $y_{n+2}=y_n$. The sequence thus consists of two terms $y_1$ and $y_2=-\beta/y_1$ repeated indefinitely.

If $\alpha\neq0$, let $|k_2|>|k_1|$; then $|k_1/k_2|^n\to0$ and $y_n\to k_2$. We thus have the

THEOREM. *If* $\alpha\neq0$, $\alpha^2>4\beta$ *and* $y_1\neq k_1$, *then*

$$v_{n+1} = \alpha - \frac{\beta}{y_n} \to k_2$$

*the root of* $k^2-\alpha k+\beta=0$ *of greater absolute value. Moreover if* $\alpha\neq k_1$, *the infinite continued fraction*

$$\alpha - \frac{\beta}{\alpha-}\,\frac{\beta}{\alpha-}\cdots \to k_2. \tag{10}$$

For example, with $\alpha=1$, $\beta=-1$,

$$y_n \to \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{1}{1+}\,\frac{1}{1+}\cdots$$

if $y_1\neq\frac{1}{2}(1-\sqrt{5})$.

*Case* 2. $\alpha^2=4\beta$, $k_1=k_2=\frac{1}{2}\alpha$.

The general solution of (2) is, from (7),

$$y_n = k_1\frac{C_1 + C_2(n+1)}{C_1 + C_2n}. \tag{11}$$

Evidently $y_n\to k_1$ irrespective of the value of $y_1$ and the continued fraction (10) converges to $\alpha/2$.

For example, with $\alpha=\beta=4$,

$$y_{n+1} = 4 - \frac{4}{y_n} \to 2.$$

*Case* 3. $\alpha^2<4\beta$, $k_1$, $k_2$ *conjugate complex.*

The general solution of (2) is still given by (8); and if we write

$$k_1 = r(\cos\theta + i\sin\theta), \qquad k_2 = r(\cos\theta - i\sin\theta),$$

it takes the form

$$y_n = r\,\frac{C_1\cos(n+1)\theta + C_2\sin(n+1)\theta}{C_1\cos n\theta + C_2\sin n\theta}. \tag{12}$$

From (5) we have

$$k_1k_2 = r^2 = \beta, \qquad k_1 + k_2 = 2r\cos\theta = \alpha;$$

hence we may take

$$r = \sqrt{\beta}, \qquad \theta = \cos^{-1}\frac{\alpha}{2\sqrt{\beta}} \qquad (0 < \theta < \pi). \tag{13}$$

If we choose $\gamma$ in the interval $-\frac{1}{2}\pi < \gamma \leqq \frac{1}{2}\pi$ so that

$$\frac{\cos\gamma}{C_1} = \frac{\sin\gamma}{C_2} \quad \text{or} \quad \gamma = \tan^{-1}\frac{C_2}{C_1},$$

(12) becomes

$$y_n = r\,\frac{\cos(n\theta + \theta - \gamma)}{\cos(n\theta - \gamma)} \tag{14}$$

in which $\gamma$ plays the role of arbitrary constant. In particular when $\gamma = 0$ $(C_2 = 0)$,

$$y_n = r\,\frac{\cos(n+1)\theta}{\cos n\theta}, \qquad y_1 = r\,\frac{\cos 2\theta}{\cos\theta} = \alpha - 2\,\frac{\beta}{\alpha};$$

and when $\gamma = \pi/2$ $(C_1 = 0)$,

$$y_n = r\,\frac{\sin(n+1)\theta}{\sin n\theta}, \qquad y_1 = r\,\frac{\sin 2\theta}{\sin\theta} = \alpha.$$

For values of $\gamma$ other than 0 or $\pi/2$ we write (14) in the form

$$y_n = r\cos\theta - r\sin\theta\tan(n\theta - \gamma) \tag{15}$$

and consider the subcases when $\theta/\pi$ is rational or irrational.

*Subcase* 3.1. $\theta/\pi = p/q$, *rational.*

We assume that $p$ and $q$ are coprime. Then since

$$\tan(q\theta + \theta - \gamma) = \tan(p\pi + \theta - \gamma) = \tan(\theta - \gamma),$$

we see that $y_{q+1} = y_1$. Thus the sequence contains only $q$ distinct terms $y_1, y_2, \cdots, y_q$, which repeat indefinitely in this order. The sequence has $q$ cluster points and tends to no limit.

For example, the sequences $\{y_n\}$ defined by

$$y_{n+1} = 1 - \frac{1}{y_n}, \qquad y_{n+1} = 2 - \frac{2}{y_n}, \qquad y_{n+1} = 3 - \frac{3}{y_n}$$

have the periods 3, 4, 6 respectively.

*Subcase 3.2.* $\theta/\pi = \lambda$ *irrational.*

All terms of the sequence are now distinct. For $y_m = y_n$ implies that

$$\tan(m\theta - \gamma) = \tan(n\theta - \gamma), \qquad (m-n)\theta = N\pi,$$

where $N$ is an integer, and $\theta/\pi$ would be rational.

When $\theta/\pi = p/q$, $\{y_n\}$ has $q$ cluster points. We now show that when $\theta/\pi$ is irrational, $\{y_n\}$ has every point of the real continuum as cluster point. To this end we appeal to

KRONECKER'S THEOREM. *If* $\lambda$ *is irrational, the set of points* $x_n = n\lambda - [n\lambda]$ *is dense in the interval* (0, 1); *that is, all points of* (0, 1) *are cluster points of the set.**

*Proof.* With $\theta = \lambda\pi$, $\gamma = c\pi$,

$$\tan(n\theta - \gamma) = \tan(n\lambda - c)\pi = \tan(n\lambda - [n\lambda] - c)\pi = \tan(\xi_n - c)\pi$$

where the set $\xi_n = n\lambda - [n\lambda]$ is dense in the interval (0, 1) and $c$ is a fixed number between $-\frac{1}{2}$ and $\frac{1}{2}$. Hence the point set $(\xi_n - c)\pi$ is dense in the interval $(-c\pi, (1-c)\pi)$; and since this interval is of length $\pi$, the set $x_n = \tan(\xi_n - c)\pi$ is dense in the interval $(-\infty, \infty)$.

Now $\theta \neq 0$ or $\pi/2$ and hence in (15) $\sin\theta$ and $\cos\theta$ are both nonzero. Thus the point set $x_n$ generates another

$$y_n = r\cos\theta - r\sin\theta\, x_n$$

which is dense in the interval $(-\infty, \infty)$. In other words, every point of the real continuum is a cluster point of the sequence $\{y_n\}$.

We state these results in the

THEOREM. *When* $\alpha$, $\beta$ *are real,* $\alpha^2 > 4\beta$, *and* $\theta = \cos^{-1}\alpha/2\sqrt{\beta}$, *the sequence* $\{y_n\}$ *defined by*

$$y_{n+1} = \alpha - \frac{\beta}{y_n}, \qquad y_1 \text{ real},$$

*has a finite or infinite number of cluster points according as* $\theta/\pi$ *is rational or irrational. More specifically,*

(a) *when* $\theta/\pi = p/q$ *a rational fraction in its lowest terms, the sequence takes on just* $q$ *distinct values* $y_1, y_2, \cdots, y_q$, *which are thereafter repeated in this order;*

(b) *when* $\theta/\pi$ *is irrational, the terms of the sequence form a dense set in the entire real continuum.*

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* Hardy and Wright, *The Theory of Numbers*, Oxford, 1938, Theorem 439, p. 364.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 1176. *Proposed by J. L. Brenner, State College of Washington*

Given $a$, $b$, $c$, $d$ real and $ad-bc=1$. Show that

$$Q = a^2 + b^2 + c^2 + d^2 + ac + bd \neq 0, 1, -1.$$

E 1177. *Proposed by W. R. Utz, University of Missouri*

Describe the three types of plane loci of points the product of whose distances from a point and a line is constant.

E 1178. *Proposed by A. J. Goldman, Princeton University*

Prove that there exists a positive constant $c$ with the following property: If $T$ is any triangle whose area exceeds $c$, then the product of the lengths of the sides of $T$ is greater than the area of $T$. What is the best possible value of $c$?

E 1179. *Proposed by C. D. Olds, San Jose State College*

Let $f$ be an operator such that $f(z)=(|z|+z)/2$, and define $f^2(z)=f\{f(z)\}$, $\cdots$, $f^n(z)=f\{f^{n-1}(z)\}$. Evaluate

$$\lim_{n\to\infty} f^n(i),$$

where $i=\sqrt{-1}$.

E 1180. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

### SOLUTIONS

#### Pythagorean Parallelopipeds

E 1146 [1955, 40]. *Proposed by P. B. Johnson, Occidental College*

Show that any rectangle whose edges and diagonal are measured in integers can be made the base of a rectangular parallelopiped whose three edges and main diagonal are measured in integers.

*Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.* Let $c$ be the diagonal of the rectangle. If $c$ is odd, take $(c^2-1)/2$ as the integral height and the integral main diagonal will be $(c^2+1)/2$; if $c$ is even, take $(c^2-4)/4$ as the integral height and the integral main diagonal will be $(c^2+4)/4$.

Also solved by W. B. Carver, R. L. Caskey, G. B. Charlesworth, R. J. Cormier, Hüseyin Demir, Monte Dernham, Fred Discepoli, E. I. Gale, Michael Goldberg, William Googe, Vern Hoggatt, R. T. Hood, J. M. Howell, A. R. Hyde, Blair Kinsman, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, L. V. Mead, T. F. Mulcrone, C. S. Ogilvy, Sidney Penner, Walter Penney, J. V. Pennington, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., J. R. Slagle, D. R. Sudborough, A. V. Sylwester, R. O. Virts, Chih-yi Wang, and the proposer. Late solution by F. W. Saunders.

The proposer pointed out that we have here an aid to the problem of finding easy teaching examples—in this case vectors whose direction cosines are rational. H. W. Becker called attention to generalizations of the problem appearing in Dickson's *History of the Theory of Numbers*, vol. II, p. 319 and p. 509, and *Mathematics Magazine*, vol. 28, 1955, p. 154. Ogilvy wondered if it is possible for either or both of the other face diagonals to be integers also. In this connection see E 525 [Feb. 1943].

### A Set of Integral Triangles

E 1147 [1955, 40]. *Proposed by E. P. Starke, Rutgers University*

If $\cos\alpha$ is rational $(0<\alpha<\pi)$, prove there are infinitely many triangles with integer sides having $\alpha$ as one angle. In particular, given $\cos\alpha=r/s$, find a three-parameter solution for the sides $a, b, c$.

*Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* By the law of cosines

$$s(c+a)(c-a) = b(2cr-bs).$$

This will be satisfied if

$$ms(c+a) = n(2cr-bs) \quad \text{and} \quad n(c-a) = mb,$$

or if

$$na + mb - nc = 0 \quad \text{and} \quad msa + nsb + (ms-2nr)c = 0.$$

It follows that

$$a = ts(m^2+n^2) - 2tmnr, \quad b = 2tn(nr-ms), \quad c = ts(n^2-m^2).$$

This problem has been solved previously by Züge, *Archiv Math. Phys.* (2) 17, 1900, 354. See Dickson, *History of the Theory of Numbers*, vol. II, p. 215.

Also solved by W. J. Blundon, W. B. Carver, Hüseyin Demir, Michael Goldberg, D. C. B. Marsh, W. V. Parker, Walter Penney, J. V. Pennington, R. J. Wisner, and the proposer.

**Two Equiareal Triangles**

E 1148 [1955, 40]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $a$, $b$, $c$ be arbitrary points on the sides $BC$, $CA$, $AB$ of triangle $ABC$, and let $A'$, $B'$, $C'$ be the reflections of $A$, $B$, $C$ in the midpoints of the segments $bc$, $ca$, $ab$. Show that triangles $abc$ and $A'B'C'$ have equal areas.

*Solution by Hüseyin Demir, Zonguldak, Turkey.* Let $a'$, $b'$, $c'$ be the reflections of $a$, $b$, $c$ in the midpoints of $BC$, $CA$, $AB$. Since, by a well known property, $abc$ and $a'b'c'$ have equal areas, we shall prove that $a'b'c'$ and $A'B'C'$ have equal areas. From $\overrightarrow{aB'}=\overrightarrow{Bc}=\overrightarrow{c'A}$, $\overrightarrow{aC'}=\overrightarrow{Cb}=\overrightarrow{b'A}$ we get $b'c'=B'C'$. Similarly $c'a'=C'A'$, $a'b'=A'B'$, and triangles $a'b'c'$ and $A'B'C'$ are actually congruent.

Also solved by W. B. Carver, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. F. Pinzka, Roscoe Woods, and the proposer.

Pinzka called attention to two similar results in R. A. Johnson, *Modern Geometry* (1929), p. 80. Carver, Hyde, Ogilvy, and Woods gave simple solutions using oblique coordinates.

*Editorial Note.* The above solution shows that triangles $a'b'c'$, $A'B'C'$ are not only congruent, but also homothetic. It follows that if $a$, $b$, $c$ are collinear on a line $L$, then $A'$, $B'$, $C'$ are also collinear on a line $L'$ parallel to the reciprocal transversal of $L$. Consequently, if $L$ is a Simson line of triangle $ABC$, then $L$ and $L'$ are perpendicular.

**First Perfect Square After $2^n$**

E 1149 [1955, 40]. *Proposed by A. S. Gregory, University of Illinois*

For each $n=1, 2, \cdots$ find the least positive integer which when added to $2^n$ yields a perfect square.

*Solution by Michael Goldberg, Washington, D. C.* Let $2^n+c=a^2$. For the smallest positive $c$, $a^2$ must be taken as the next square larger than $2^n$. Then

$$c = a^2 - 2^n = ([2^{n/2}] + 1)^2 - 2^n.$$

When $n=2k$, $c=(2^k+1)^2-2^{2k}=2^{k+1}+1$. When $n\neq 2k$, the expression does not simplify.

Also solved by Leon Bankoff, W. B. Carver, Fred Discepoli, I. A. Dodes, A. J. Goldman, A. R. Hyde, M. S. Klamkin, L. J. Lange, D. C. B. Marsh, P. A. Piza, Azriel Rosenfeld, C. M. Sandwick, Sr., D. R. Sudborough, A. V. Sylwester, K. B. Williams, and the proposer. Some of these solutions were not complete. Late solution by Hüseyin Demir.

The proposer considered the more general problem: Let $A$, $B$ be fixed integers greater than 1. For each $n=1, 2, \cdots$ find the least positive integer which when added to $A^n$ yields a perfect $B$th power.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4648. *Proposed by W. S. Loud, University of Minnesota*

A power series, $\sum_{n=0}^{\infty} a_n z^n$, has radius of convergence unity. It is known to converge at every point of its circle of convergence. Does it necessarily converge absolutely on the circle of convergence?

4649. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Does the fact that a power series has uniformly bounded partial sums on the circle $|z|=1$ necessarily imply that it converges everywhere on $|z|=1$?

4650. *Proposed by J. L. Massera, Instituto de Matematica Y Estadistica, Montevideo, Uruguay*

Let $p_1, \cdots, p_k$ be any fixed indices, $0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_k < n$ and let $D(x_1, \cdots, x_k)$ be the $n$th order determinant whose elements $a_{ij}$ are defined by: $a_{ij}=x_h$ if $i+j=p_h+1$, $h=1, \cdots, k$; $a_{ij}=-x_h$ if $i+j=n+p_h+1$, $h=1, \cdots, k$; $a_{ij}=0$ otherwise. Show that $D$, considered as a function of the real variables $x_1, \cdots, x_k$, may change its sign only in the neighborhood of those values which satisfy one of the relations $x_1 \pm x_2 \pm \cdots \pm x_k = 0$.

4651. *Proposed by the late Joseph Rosenbaum.*

On the sides of a parallelogram $A_1A_2A_3A_4$, equilateral triangles $A_iA_{i+1}B_i$ are constructed exteriorly. Then equilateral triangles $B_iB_{i+1}C_i$ are constructed interiorly to $B_1B_2B_3B_4$. Prove that $C_1C_2C_3C_4$ coincides with $A_1A_2A_3A_4$.

Generalize to the case where the original parallelogram is replaced by a polygon of $n$ sides.

4652. *Proposed by Albert Wilansky, Lehigh University*

Let $f$ be a Lebesgue integrable function on an interval. Let $t_n$ be the measure of the set of points $x$ such that $|f(x)|>n$. Prove that $\sum t_n < \infty$.

## SOLUTIONS

### An Improper Integral

4588 [1954, 350]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Let $n>1$. Evaluate

$$\int_0^\infty (\sqrt[n]{x^n+1}-x)dx.$$

*Solution by R. A. Rosenbaum, Wesleyan University.* The limit tests show that the given integral diverges for $n\leqq 2$ and converges for $n>2$. For $n>2$, call the value of the integral, $I$. Then integration by parts gives

$$2I=\int_0^\infty (x^n+1)^{1/n-1}dx.$$

The substitution $x^n+1=1/y$ leads to the beta-function

$$I=\frac{1}{2n}\int_0^1 y^{-2/n}(1-y)^{1/n-1}dy=\frac{1}{2n}\,\mathrm{B}\left(1-\frac{2}{n},\frac{1}{n}\right).$$

Also solved by Ranko Bojanić, Leonard Carlitz, R. V. Esperti, H. E. Fettis, Martin Kruskal, Viktors Linis, A. E. Livingston, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

### Separable Metric Spaces

4589 [1954, 350]. *Proposed by Casper Goffman, Wayne University*

It is well known that the interval (0, 1) is the union of a set of the first category and a set of measure zero. Generalize this result to arbitrary separable metric spaces.

*Solution by the Proposer.* A generalization is furnished by the Theorem: *If $S$ is a separable metric space, then $S=E\cup Z$, where $E$ is of the first category in $S$ and $Z$ is zero dimensional in the Hausdorff sense.* This is the required generalization since every Hausdorff zero dimensional set in (0, 1) has Lebesgue measure zero.

To prove this theorem, let $a_1, a_2, \cdots, a_n, \cdots$ be dense in $S$. For every $n$ and $m$ let $\sigma_{nm}$ be a sphere of center $a_n$ and radius $(m\cdot 2^{nm})^{-1}$. Let

$$T_m=\bigcup_{n=1}^{\infty}\sigma_{nm},\qquad T=\bigcap_{m=1}^{\infty}T_m.$$

(a) For every $m$, $S-T_m$ is nowhere dense in $S$, so that $S-T$ is of the first category.

(b) Let $\epsilon>0$. An easy calculation shows that for every $0<\eta<\epsilon$, $T$ may be covered by a sequence of spheres $\sigma_1, \sigma_2, \cdots, \sigma_n, \cdots$ whose radii $r_1, r_2, \cdots,$

$r_n, \cdots$ are all less than $\eta$, such that $\sum_{n=1}^{\infty} r_n^{\epsilon} < \eta$, whence $T$ is of Hausdorff $\epsilon$-dimensional measure zero. In other words, $T$ is a Hausdorff zero dimensional set.

**Divisors of the Fermat Numbers**

4590 (1954, 350). *Proposed by Paul Erdös, University of Notre Dame*

Fermat's conjecture that all numbers of the form

$$F_n = 2^{2^n} + 1$$

are prime was proved wrong by Euler. Show, however, that $\sum 1/d \to 0$ as $n \to \infty$, where $d$ ranges over all the divisors of $F_n$ except 1.

*Solution by Ranko Bojanić, Mathematical Institute, Belgrade.* Since

$$\sum_{d|n} \frac{1}{d} - 1 = \frac{\sigma(n)}{n} - 1,$$

where $\sigma(n)$ is the sum of all divisors of $n$, including 1 and $n$, the required result is equivalent to the statement that

$$\frac{\sigma(F_n)}{F_n} \to 1, \qquad n \to \infty. \tag{1}$$

We have first, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$,

$$1 > \frac{n}{\sigma(n)} = \prod_{p|n} \frac{1 - p^{-1}}{1 - p^{-\alpha-1}} \geqq \prod_{p|n} (1 - p^{-1}). \tag{2}$$

Now let $p$ denote a prime divisor of $F_n$ $(2 < p \leqq F_n)$. Then $p$ is of the form $2^{n+1}k + 1$ and hence

$$p > 2^n. \tag{3}$$

From (2) and (3) it follows that

$$1 > \frac{F_n}{\sigma(F_n)} \geqq (1 - 2^{-n})^{\omega},$$

where $\omega$ is the number of different prime factors of $F_n$. Since

$$F_n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega}^{\alpha_{\omega}} \geqq 2^{(\alpha_1+\alpha_2+\cdots+\alpha_{\omega})n} \geqq 2^{\omega n},$$

we have

$$\omega n \leqq \frac{\log (2^{2^n} + 1)}{\log 2} = 2^n + \frac{\log (1 + 2^{-2^n})}{\log 2},$$

or $\omega \leqq 2^n/n + 1$. Hence

$$1 > \frac{F_n}{\sigma(F_n)} \geqq (1 - 2^{-n})^{1+2^n/n} \to 1, \qquad n \to \infty,$$

and so (1) is proved.

We observe finally that from (1) follows immediately

$$\frac{\phi(F_n)}{F_n} \to 1, \qquad n \to \infty,$$

where $\phi(n)$ is the number of integers not exceeding $n$ and prime to $n$.

The same results hold also for the numbers of the form $M_p = 2^p - 1$, where $p$ is a prime: we have

$$\frac{\sigma(M_p)}{M_p} \to 1, \qquad \frac{\phi(M_p)}{M_p} \to 1, \qquad p \to \infty.$$

Also solved by Armand Brumer, and Leo Moser and J. Lambek (jointly).

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## RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

*All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio*

*Introduction to Modern Algebra and Matrix Theory.* By R. A. Beaumont and R. W. Ball. New York, Rinehart and Company, Inc., 1954. 12+331 pages. $6.00.

This is a carefully written introduction to the concepts and techniques of modern algebra, with particular emphasis on the theory of matrices. The topics covered are the following: Chapter I: elementary aspects of matrices and determinants; Chapter II: groups of transformations; Chapter III: vector spaces; Chapter IV: elementary group theory; Chapter V: rings and fields; Chapter VI: polynomials and the algebraic theory of fields; Chapter VII: matrices with polynomial elements; Chapter VIII: canonical forms.

This book is designed mainly for the intelligent third and fourth year undergraduate, whose major interest is in mathematics, the physical sciences, or certain of the social sciences. Considerable emphasis is placed on rigor, and the average student will find more than enough material here to occupy his time and tax his ability. A number of exercises are included in an effort to make the abstractions more meaningful. The material has been arranged so that a one-semester course on the theory of matrices can be given independently. Such a

course would be based on Chapters I, II, III, VII, and VIII, and need not stress abstractions.

Matrix theory and abstract algebra have become an integral part of every well-rounded undergraduate mathematics curriculum. This has resulted in the publication in recent years of a number of worthwhile accounts. It is not our purpose here to compare each of the available texts with the one by Beaumont and Ball. The better ones will doubtless compete with each other, and anyone planning a course along these lines is urged to look at all of the available material with considerable care before selecting the book best suited to his particular needs.

H. J. RYSER
The Ohio State University

*Universal Mathematics*. By the 1954 Summer Writing Group.

Part I of this "book of experimental text materials" was published in September 1954 by the University of Kansas Book Store. Part II will be available in September 1955 as a publication of the Tulane University Book Store. The writing of these lithographed books has been an activity of the Association's Committee on the Undergraduate Mathematical Program. Persons wishing examination copies of either part should communicate with Professor G. B. Price, Mathematics Department, University of Kansas, Lawrence, Kansas.

C.B.A.

## NEW BOOKS RECEIVED

*Transform Calculus with an Introduction to Complex Variables*. By E. J. Scott. New York, Harper and Brothers, Publishers, 1955. 8+330 pages. $7.50.

*Numerical Methods*. By A. D. Booth. New York, Academic Press, Inc., 1955. 7+195 pages. $6.00.

*An Introduction to Stochastic Processes*. By M. S. Bartlett. New York, Cambridge University Press, 1955. 14+312 pages. $6.50.

*Binomial Coefficients*. Edited by J. C. P. Miller. (Vol. 3 of the Royal Society Mathematical Tables). New York, American Branch of Cambridge University Press, 1954. 8+162 pages. $5.50.

*Proceedings of the First Conference on Training Personnel for the Computing Machine Field*. Edited by Arvid W. Jacobson. Detroit, Michigan, Wayne University Press, 1955. 104 pages. $5.00.

*Advanced Mathematics for Engineers*, Third Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc., 1955. 14+548 pages. $6.50.

*First Course in Algebra for Colleges*. By L. J. Adams, New York, Henry Holt and Company, 1955. 6+217 pages. $3.00.

*Integers and Theory of Numbers*. By A. A. Fraenkel. New York, Scripta Mathematica, 1955. 102 pages. $2.75.

*The Theory of Numbers*. By B. W. Jones. New York, Rinehart and Company, 1955. 11+143 pages. $3.75.

*College Algebra*. By P. R. Rider. New York, The Macmillan Company, 1955. 14+397 pages. $4.00.

*Slide Rule Operations*. By D. R. Sudborough and H. W. Zeoli. Ann Arbor, Michigan, J. W. Edwards, Publisher, 1954. 76 pages. $1.50.

*Business Mathematics*. By J. A. Mira and George Hartmann. New York, D. Van Nostrand Company, 1955. 6+341 pages. $4.50.

*Higher Transcendental Functions*, Vol. 3. Sponsored by California Institute of Technology, The Bateman Project Staff. Edited by A. Erdelyi. New York, McGraw-Hill Book Company, 1955. 17+292 pages.

*General Topology*. By J. L. Kelley. New York, D. Van Nostrand Company, 1955. $8.75.

*Plane Trigonometry*. By C. R. Wylie, Jr., New York, McGraw-Hill Book Company, 1955. 381 pages. $4.00.

*Parmi Les Belles Figures de la Geometrie dans l'espace*. By Victor Thébault. Paris, Librairie Vuibert, 63, Boulevard Saint-Germain, 1953. 286 pages.

*College Algebra, 4th Edition*. By W. L. Hart. Boston, D. C. Heath and Company, 1953. 10+480 pages.

*Curso de Analisis Matematico, Tomo III*. By Cristobal de Losada y Puga. Lima, Peru, 1954, Imprenta Santa Maria, Calle de Santa Catalina, 661. 814 pages.

*Tables of Sines and Cosines for Radian Arguments*. By U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, 43. Washington, D. C., 1955. 278 pages. $3.00.

*Analytic Geometry, 2nd Edition*. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1955. 9+310 pages. $3.75.

*Analytic Geometry*. By N. H. McCoy and R. E. Johnson. New York, Rinehart and Company, Inc., 1955. 14+301 pages. $4.00.

*Calculus*. By W. L. Hart. Boston, D. C. Heath and Company, 1955. 13+626 pages. $5.50.

*Transactions of Symposia in Applied Mathematics, Vol. 2*. Symposium on Computing Mechanics, Statistics and Partial Differential Equations, held at the University of Chicago, Apr. 29–30, 1954. New York, Interscience Publishers, Inc., 1954. 216 pages. $5.00.

*Plane Algebraic Curves*. By E. J. F. Primrose. New York, St. Martin's Press, Inc., 103 Park Ave., New York 17, 1955. 7+111 pages. $3.00.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### MEMORIAL TO PROFESSOR E. L. POST

A memorial to the late Professor Emil L. Post is being established at The City College of New York. The form of the memorial will depend to some extent on the response to the project. Any who so desire may send contributions to Professor H. P. Wirth, Chairman of the Memorial Committee, or to Professor B. P. Gill, Treasurer, The City College, New York 31, New York.

### PERSONAL ITEMS

Professor V. O. McBrien of the College of the Holy Cross was official delegate of the Association at the inauguration of President A. B. Bronwell of Worcester Polytechnic Institute on April 30, 1955.

Professor J. I. Tracy of Texas Christian University was the official representative of the Association at the inauguration of President W. M. Tate of Southern Methodist University on May 5, 1955.

Dr. R. C. Blanchfield, National Research Council Fellow at Massachusetts Institute of Technology, has been awarded a National Science Foundation Postdoctoral Fellowship for 1955–1956.

Dr. Claude Chevalley of Columbia University has been awarded a Guggenheim Fellowship.

Assistant Professor Edwin Halfar of the University of Nebraska was awarded a research grant-in-aid by the University Research Council for summer work.

Dr. W. C. Hamilton, formerly a graduate fellow at the Crellin Laboratory of Chemistry, California Institute of Technology, is now a National Science Foundation Post-doctoral Fellow at the Mathematical Institute, Oxford, England.

Professor M. Gweneth Humphreys, chairman of the Department of Mathematics of Randolph-Macon Woman's College, is a recipient of a Faculty Fellowship awarded by the Fund for the Advancement of Education. Dr. Humphreys, who is on sabbatical leave from the College during 1955–56, is studying at the University of British Columbia. Also, Dr. Humphreys plans to study through short visits the undergraduate programs in several colleges and universities.

Professor Marston Morse of the Institute for Advanced Study was awarded the honorary degree of Doctor of Science by the University of Maryland on March 25, 1955.

Professor Ivan Niven of the University of Oregon has been awarded a Faculty Fellowship by the Fund for the Advancement of Education and is on leave of absence at the University of California, Berkeley.

Professor H. B. Ribeiro of the University of Nebraska was awarded a research grant-in-aid by the University Research Council for summer work.

Dr. W. R. Wasow of the University of California at Los Angeles has been awarded a Fulbright fellowship and is on leave of absence in Rome.

Associate Professor G. W. Whitehead of Massachusetts Institute of Technology is on leave of absence for the year 1955–1956 on a Guggenheim Fellowship at Oxford University and at the University of Paris, France.

Brown University announces the following: Mr. Paul Slepian, formerly an assistant at the University, and Dr. Heini Halberstam of University College of the Southwest of England have been appointed to instructorships; Associate Professor W. S. Massey has been promoted to a professorship; Assistant Professor F. M. Stewart has been promoted to an associate professorship; Professor Bjarni Jonsson is on leave of absence for the year 1955–1956 and is spending the year at the University of California as Visiting Associate Professor.

Dartmouth College announces that it is inaugurating a new program for gifted mathematics students.

Fresno State College reports: Professor F. R. Morris, formerly chairman of the Department of Mathematics and head of the Physical Science Division, has retired; Associate Professor Roy Dubisch has been promoted to a professorship and appointed Chairman of the Department of Mathematics, Assistant Professors John Christopher of Pacific University and R. D. Stalley of Iowa State College have been appointed to instructorships.

Illinois Institute of Technology announces: Assistant Professor Alfonso Shimbel of the University of Chicago and Dr. C. A. Nicol of the University of Texas have been appointed to instructorships.

Knox College reports the following: Professor Rothwell Stephens, head of the Department of Mathematics, has been appointed Hitchcock Professor of Mathematics; Dr. W. C. Ross, Jr., of the State University of Iowa and Mr. Dwain Small, formerly a teacher at Richmond, Indiana, have been appointed to instructorships.

At Massachusetts Institute of Technology: Associate Professor A. P. Calderon of Ohio State University and the Institute for Advanced Study has been appointed to an associate professorship; Assistant Professor N. C. Ankeny of Johns Hopkins University and Dr. L. N. Howard, recently Higgins Lecturer at Princeton University, have been appointed to assistant professorships; Dr. A. P. Mattuck, National Science Foundation Fellow at Harvard University, and Dr. W. F. Reynolds, previously an instructor at the College of the Holy Cross and research fellow at Harvard University, have been appointed C. L. E. Moore Instructors; Dr. Hartley Rogers, Jr., formerly Benjamin Peirce Instructor at Harvard University, and Dr. J. J. Levin of Purdue University have been appointed Visiting Lecturers; Assistant Professor Kenkichi Iwasawa has been promoted to an associate professorship; Professor Raphael Salem has been appointed to a professorship at the Sorbonne, France; Professor Norbert Wiener

is on leave of absence for the year 1955–1956 and will spend most of the time in India.

At the University of Wisconsin, a conference entitled "The Computing Laboratory in the University" was held August 17–19, 1955.

Washington University announces: Associate Professor I. I. Hirschman, Jr., has been promoted to a professorship; Assistant Professor H. Margaret Elliott has been promoted to an associate professorship.

Wayne University reports the following: Associate Professor W. F. Eberlein of the University of Wisconsin has been appointed Visiting Professor for 1955–1956; Assistant Professor S. I. Goldberg of Lehigh University has been appointed to an assistant professorship; Mr. Robert Kuller, formerly an instructor at Dartmouth College, and Mrs. Patricia J. Wells, recently an assistant at Michigan State University, have been appointed to instructorships; Associate Professor Benjamin Epstein is on sabbatical leave for 1955–1956 and is at Stanford University; Associate Professor Y. W. Chen, while on leave of absence for the first semester, is at the Institute for Advanced Study; Miss Winifred Burroughs has been appointed Visiting Lecturer at Wheaton College, Massachusetts.

Mr. W. R. Allen, formerly a mathematician at the University of Chicago, Institute for Air Weapons Research, has a position as an associate at the Forrestal Research Center, Princeton University.

Assistant Professor F. J. Arena of North Dakota State College has been appointed to an assistant professorship at Canisius College.

Mr. P. H. Arnold, previously a staff member at the Sandia Corporation, Albuquerque, New Mexico, is now a computing analyst at North American Aviation, Columbus, Ohio.

Associate Professor R. W. Barnard of the University of Chicago has retired.

Associate Professor P. M. Batchelder of the University of Texas has retired.

Mr. R. E. Bayles, formerly an actuarial assistant for John Hancock Mutual Life Insurance Company, Boston, Massachusetts, is at the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. R. V. Benson has been appointed to an instructorship at Los Angeles City College.

Dr. W. J. Berger, recently an assistant at Carnegie Institute of Technology, has a position as a mathematician with R.C.A. Service Company, Patrick Air Force Base, Cocoa, Florida.

Assistant Professor R. R. Bernard of Yale University has been appointed to an associate professorship at Davidson College.

Professor R. W. Brink of the University of Minnesota is now Acting Assistant Dean of the College of Science, Literature and the Arts.

Dr. Eleazer Bromberg, previously assistant chief of the Computing Center, Institute of Mathematical Sciences, New York University, has been appointed to an associate professorship in the Graduate School of Arts and Sciences, New York University.

Assistant Professor J. L. Brown, Jr., of Pennsylvania State University has been promoted to an associate professorship.

Associate Professor R. H. Bruck of the University of Wisconsin has been promoted to a professorship.

Dr. R. J. Buehler, formerly a staff member of the Sandia Corporation, Albuquerque, New Mexico, is a project associate at the Naval Research Laboratory, University of Wisconsin.

Associate Professor Hobart Bushey of Hunter College has been promoted to a professorship.

Reverend A. E. Cahill, instructor at Belmont Abbey College, has been promoted to an assistant professorship.

Professor W. L. Carter of the Territorial College of Guam has been appointed to an assistant professorship at the University of Cincinnati.

Mr. J. T. Clausen, Jr., chief of Methods and Data Analysis, White Sands Proving Ground, Las Cruces, New Mexico, has accepted a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. J. W. Coy of Michigan State University has a position as an analytic statistician at White Sands Proving Ground, Las Cruces, New Mexico.

Assistant Professor Avron Douglis of New York University has been promoted to an associate professorship.

Professor W. L. Duren, Jr., of Tulane University has been appointed Professor of Mathematics and Dean of the College of Arts and Sciences, University of Virginia.

Dr. L. K. Durst of Rice Institute has been promoted to an assistant professorship.

Mr. J. S. Elston, associate actuary at Travelers Insurance Company, Hartford, Connecticut, has retired.

Mr. D. O. Etter, formerly a graduate student at Tulane University, has accepted a position as a chemist in the Control Laboratory, International Lubricant Corporation, New Orleans, Louisiana.

Dr. A. G. Fadell of the University of Buffalo has been promoted to an assistant professorship.

Mr. A. M. Fleishman, recently a mathematician at the United States Naval Proving Ground, Dahlgren, Virginia, has a position as an engineer for the Radio Corporation of America, Camden, New Jersey.

Mr. G. C. Francis, previously a lecturer at Columbia University, is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Dr. José Gallego-Diaz of Escuela Especial de Ingenieros Agronomos, Madrid, Spain, has been named Professor of General Physics.

Dr. T. M. Gallie, Jr., formerly a research instructor at Duke University, is now a research engineer for Humble Oil and Refining Company, Houston Research Center, Texas.

Mr. S. I. Gass, formerly a mathematician with the Computation Division, Headquarters, United States Air Force, Washington, D. C., has accepted a position as an applied science representative with the I.B.M. Corporation, Washington, D. C.

Mr. A. L. Gilmore, Jr., previously a radar instructor at Keesler Air Force Base, Biloxi, Mississippi, has accepted a position as a mathematician at Air Proving Ground, Eglin Air Force Base, Florida.

Assistant Professor Seymour Ginsburg of the University of Miami has a position as a research analyst at Northrop Aircraft Corporation, Hawthorne, California.

Mr. M. L. Goldwater, recently an engineer for Librascope, Glendale, California, is now a research engineer for J. B. Rea, Santa Monica, California.

Mr. R. M. Gordon, formerly a mathematical analyst for Lockheed Aircraft Corporation, Burbank, California, is now an electronic applications specialist and instructor in programming with National Cash Register Company, Dayton, Ohio.

Mr. W. T. Gregorzak, previously a teaching assistant at Rutgers University, has been appointed Research Staff Assistant at Johns Hopkins University Radiation Laboratory, Baltimore, Maryland.

Mr. J. R. Hadley, formerly a manufacturing control manager for Richardson Company, Indianapolis, Indiana, is employed as a chief industrial engineer by Refrigeration Appliances, Chicago, Illinois.

Professor E. E. Haskins, head of the Department of Physics of Norwich University, has been appointed Chairman of the Department of Mathematics of Clarkson College of Technology.

Assistant Professor G. C. Helme of Pratt Institute has been promoted to an associate professorship.

Dr. P. S. Herwitz, previously a research associate for the Institute for Cooperative Research, Johns Hopkins University, has a position as a mathematician for the I.B.M. Corporation, Washington, D. C.

Associate Professor S. T. Hu of Tulane University has been appointed to a professorship at the University of Georgia.

Mr. J. H. Kaplan, previously a student at Temple University, is employed as a mathematician at Frankford Arsenal, Philadelphia, Pennsylvania.

Miss Carolyn E. Kappel, formerly a student at Carleton College, is a technical assistant at the Bell Telephone Laboratories, New York, New York.

Mr. Edgar Karst of Independence, Missouri, is now a computing analyst, Electronic Computer Section, Great Lakes Pipe Line Company, Kansas City, Missouri.

Dr. Maurice Kennedy, formerly a graduate student at California Institute of Technology, has been appointed Assistant Lecturer at University College, Dublin, Ireland.

Mr. R. B. Kiltie, previously a graduate student at New York University, has been appointed to an assistant instructorship at Newark College of Engineering.

Mr. G. J. Kleinhesselink, recently a research analyst for Northrop Aircraft Corporation, Hawthorne, California, is now a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Professor R. E. Langer of the University of Wisconsin is on leave of absence and has a contract with the Department of the Air Force, Air Research and Development Command.

Mr. H. D. Lechner, recently a student at the University of Kansas, has a position as an applied science representative for I.B.M. Corporation, Oklahoma City, Oklahoma.

Associate Professor Marguerite Lehr of Bryn Mawr College has been promoted to a professorship.

Dr. C. E. Lemke, previously a research associate at General Electric Company, Knolls Atomic Power Laboratory, Schenectady, New York, has a position as an engineer with the R.C.A. Victor Division of the Radio Corporation of America, Camden, New Jersey.

Mrs. Isabel S. Macquarrie, visiting lecturer at the Summer School of Mount Allison University in 1955, has been appointed to an instructorship at Wellesley College.

Professor W. G. Madow of the University of Illinois is on leave for the year 1955–1956 and is Visiting Professor at Stanford University.

Mr. Saul Mandel, formerly a student at the University of Oklahoma, has a position as a design engineer at Aerojet-General, Azusa, California.

Dr. N. M. Martin, previously a research associate at Willow Run Research Center, University of Michigan, has been appointed a member of the technical staff of Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. A. L. Mayerson, principal actuary in the New York State Insurance Department, New York City, is now in Paris on a Fulbright scholarship.

Acting Assistant Professor John McCarthy of Stanford University has been appointed to an assistant professorship at Dartmouth College.

Dr. Garner McCrossen of the University of Colorado has a position as a mathematician at Holloman Air Development Center, Holloman Air Force Base, New Mexico.

Mr. Emanuel Mehr of Brooklyn College has been appointed to the position of Associate Engineering Mathematician in the College of Engineering, New York University.

Professor T. E. Mergendahl of Tufts College is retiring with the title of Professor Emeritus.

Mr. R. B. Merrill, formerly a sales correspondent for the Eagle-Picher Company, Chicago, Illinois, is employed as a statistician by the Industry Export Service, Cincinnati, Ohio.

Associate Professor L. I. Mishoe of Morgan State College has been promoted to a professorship.

Assistant Professor D. R. Morrison of Tulane University has accepted a position as staff member at the Sandia Corporation, Albuquerque, New Mexico.

Mr. J. D. Munn of Mississippi Southern College has been promoted to an assistant professorship.

Mr. M. P. O'Donnell, lecturer at the University of Queensland, Brisbane, Australia, is on leave of absence for two years at Clare College, Cambridge, England.

Mr. L. A. Ondis, II, formerly a junior scientist at Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has been promoted to an associate scientist.

Mr. W. L. Phillips, Jr., previously a graduate student at Purdue University, is now a graduate assistant at the University.

Dr. J. H. Powell of the University of Detroit has been appointed to an assistant professorship at Western Michigan College of Education.

Professor J. F. Randolph, chairman of the Department of Mathematics of the University of Rochester, is on leave of absence for the academic year 1955–1956 as Visiting Professor at the American University, Beirut, Lebanon.

Associate Professor P. K. Rees of Louisiana State University has been promoted to a professorship.

Professor P. R. Rider of Washington University has retired with the title of Professor Emeritus.

Miss Marilyn A. Rogers, formerly a student at Carleton College, is now an engineering assistant at General Electric Company, Schenectady, New York.

Mr. W. G. Rouleau, previously a mathematician for Army Map Service, Washington, D. C., has a position as a mathematician and programmer with Engineering & Research Corporation, Riverdale, Maryland.

Associate Professor L. J. Savage of the Committee on Statistics, University of Chicago, has been promoted to a professorship.

Dr. Seymour Schuster of the Polytechnic Institute of Brooklyn has been promoted to an assistant professorship.

Dr. D. H. Shaffer of Carnegie Institute of Technology has a position as a research mathematician at Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Associate Professor C. Eucebia Shuler of the University of South Carolina has been promoted to a professorship.

Mr. R. F. Smith of Syracuse University has been appointed to an assistant professorship at the University of Vermont.

Mr. S. P. Spaulding, formerly an ordnance engineer at the Naval Torpedo Station, Newport, Rhode Island, has accepted a position as an operations research analyst at Electric Boat, Division of General Dynamics Corporation, Groton, Connecticut.

Miss Dorothy M. Swan of Monticello College has been appointed to an assistant professorship at Teachers College at Cortland, New York.

Professor J. A. Ward of the University of Kentucky has been granted a leave of absence for the year 1955–1956 to take a position as a mathematician with the Division of Technical Analysis, Holloman Air Force Base, New Mexico.

Dr. Warren Weaver, director of the Division of Natural Sciences, Rockefeller Foundation, has been promoted to Vice-President for Natural and Medical Sciences.

Dr. J. W. Weihe of the University of California has accepted a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Mr. Howard Young, recently in the United States Army, has a position as an actuarial trainee for Metropolitan Life Insurance Company, New York, New York.

Dr. J. W. Young, previously a mathematician with the I.B.M. Corporation, Atlanta, Georgia, is now a mathematician for I.B.M. Corporation, in Poughkeepsie, New York.

Assistant Professor G. C. Zader of The Citadel has been appointed Director of Admissions and Placement, Rose Polytechnic Institute.

Professor Emeritus R. C. Archibald of Brown University died on July 26, 1955. He was a charter member of the Association, and was President in 1922.

Professor Emeritus H. T. R. Aude of Colgate University died on June 2, 1955. He was a member of the Association for thirty years.

Professor J. W. Campbell of the University of Alberta died on January 23, 1955. He was a member of the Association for thirty-eight years.

Dr. A. P. Cowgill of Syracuse, New York, died on March 22, 1955.

Mr. C. H. Dennison, chemist for Archer Rubber Company, Milford, Massachusetts, died on March 15, 1955. He was a member of the Association for thirty years.

Mr. William Douglas of Courtenay, British Columbia, Canada, died on January 5, 1955.

Professor Henri Fehr of Geneva, Switzerland, died on November 2, 1954. He was Secretary of the International Commission on Mathematics Instruction and was honorary President of this Commission.

Professor H. T. Guard, head of the Department of Mathematics of Colorado Agricultural and Mechanical College, died on June 8, 1955.

Professor J. R. Kline of the University of Pennsylvania died on May 2, 1955. He was a member of the Association for thirty-five years.

Professor H. R. Phalen, head of the Department of Mathematics of the College of William and Mary, died on May 30, 1955. He was a charter member of the Association.

Professor C. H. Richardson of Bucknell University died on March 13, 1955. He was a member of the Association for thirty-seven years.

Professor Emeritus L. L. Smail of Lehigh University died on January 26, 1955. He was a member of the Association for thirty-one years.

## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1955 by a mail vote of the membership of the Association in the Sections indicated:

| | |
|---|---|
| Kansas | C. B. Read, University of Wichita |
| Missouri | F. F. Helton, Central College |
| Ohio | L. L. Lowenstein, Kent State University |
| Pacific Northwest | Ivan Niven, University of Oregon |
| Southeastern | F. W. Kokomoor, University of Florida |
| Southwestern | M. S. Hendrickson, University of New Mexico |
| Upper New York State | J. F. Randolph, University of Rochester |
| New England Region | G. B. Thomas, Jr., Massachusetts Institute of Technology |

Ballots were received from one-third of those eligible to vote in the above elections for sectional governors. The proportion of votes cast is always much higher in the sectional elections than in the annual elections for national officers. The highest percentage of votes cast was in the Kansas Section where votes were received from 46% of the members of the section.

H. M. GEHMAN, *Secretary-Treasurer*

### THE 1955 COMBINED MEMBERSHIP LIST

The Mathematical Association of America issues each year a Combined Membership List jointly with the American Mathematical Society. It is expected that the members of the Society for Industrial and Applied Mathematics will also be included in the 1955 List.

All members of the Association will receive a copy of the 1955 Combined Membership List sometime in December as one of the privileges of membership.

The office of the Association should be notified promptly of all changes in rank, position, and address which have not previously been reported. The final date for receipt of changes is October 15. Any errors in the 1954 Membership List should also be reported before October 15.

H. M. GEHMAN, *Secretary-Treasurer*

### CONTINUATION OF THE PROGRAM OF VISITING LECTURERS

The National Science Foundation has granted the Association the sum of $20,500 for the continuation of the Program of Visiting Lecturers for a period of approximately eighteen months, beginning on or about February 1, 1956. The Program will be administered by the Association's Committee on Visiting

Lecturers. The present committee consists of Professors G. B. Huff, B. W. Jones, and D. E. Richmond, Chairman. The Committee is authorized to select the lecturers and to arrange their itineraries.

## REPORT OF THE COMMITTEE ON THE UNDERGRADUATE MATHEMATICAL PROGRAM

**1. General statement.** The Committee on the Undergraduate Mathematical Program was appointed by President E. J. McShane in January 1953. Since it was directed to consider the problems of making available in our society the values of modern mathematics, the word *program* was used in its name rather than *curriculum*. The Committee was instructed to attack the problem with broader scientific and cultural objectives than could be expressed through another mere study of curriculum revision.

At the Kingston meeting in September 1953, the Committee reported to the Board of Governors that there exists a widespread dissatisfaction with the existing undergraduate program in mathematics, complemented by a remarkable unanimity of feeling about the nature of the deficiencies in the present program and the general nature of the program which should replace it. Believing that the reformation of college mathematics cannot be accomplished by the adoption of a new curriculum emanating from any committee, we recommended a widespread program of "doing" to overcome the inertia of the enormously ponderous structure which carries onward the present program with all of its deficiencies. The findings of the Committee and its recommendation to get on with the "doing" was adopted by the Board of Governors. The Committee, reorganized to include C. V. Newsom, who is also chairman of the important Joint Committee on Teacher Education in Mathematics, was charged with organizing the "doing" phase of the work.

This charge to go ahead with the much-talked-about revolution in the mathematics program is one which we would not dare accept as our own private responsibility. In all fields we must learn the ways in which mathematics can contribute to scientific and social thinking and support the existing technology. To this end we must enlist the aid of the men in all these fields who have the greatest wisdom and scientific insight; for the history of previous attempts at curricular revision in mathematics strongly suggests that they failed because these attempts were made upon a narrowly pedagogical and organizational basis, without the participation of the real intellectual leaders in mathematics and in the fields which use mathematics. In particular, we must find the men with ideas, who, by experimental writing and teaching, can beat into shape new courses and their textbooks, so that the rest of us can teach new and (we hope) better courses in mathematics. The Committee must guard against the adoption of arbitrary opinions of its own or of others in order that any new program or movement which we foster shall be true to the development of mathematics itself.

In the existing state of affairs there are a large number of conditions which bear upon any movement to modify the traditional program, whether by natural

forces or by arbitrary effort. From the scientific standpoint, one must take into account the extended mathematical needs of modern engineering and physical science. At the same time such sciences as econometrics, physiology, sociology, and genetics seem to demand, in part at least, entirely new mathematics. Moreover, the emergence of new mathematical technologies based upon high-speed digital automata presents serious problems of policy. For the intriguing gadgetry of these machines tends to obscure the potentially great magnitude of their impact upon men. This current development based upon the old subjects of symbolic logic, numerical analysis, and combinatorial theory is another example of the fact that the work of the creative mathematician tends to be applied much less directly than that of the empirical scientist and, even so, largely through the mathematical knowledge which technical workers in other fields acquire in school and college.

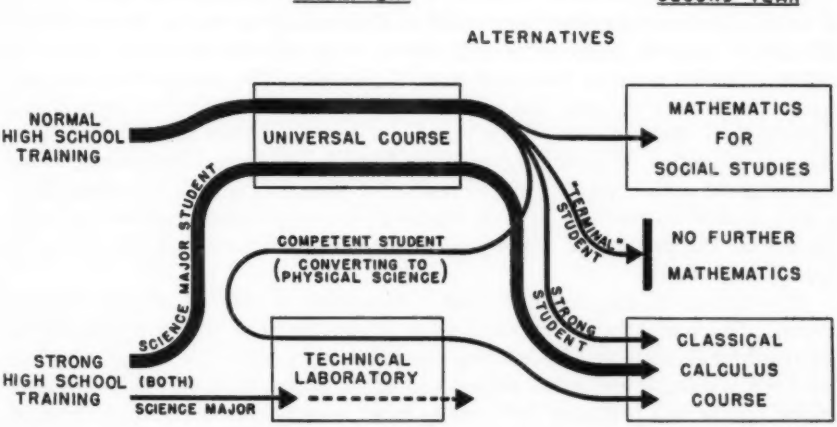
Thus, turning to the educational conditions affecting our problem, one finds that probably the central one is the desertion of elementary teaching by the best mathematicians, old and young. Besides this, there is the rigid sequential organization of the traditional program with each course depending heavily on technical prerequisites, the compartmentation in college programs with attendant difficulties for the student who transfers or changes program, the growing tendency to repeat high school courses in college, the terminal course idea, the emphasis upon cultural and liberal aspects in education today, the current popularity of logic, the recent studies of articulation of school and college, the incipient movement to train a special breed of college mathematics teachers by means of special education programs, and the complicated relationships of the policies of foundations, which might support the writing of new text material, to the publishing companies, and the royalty rights of authors. There are, of course, many mathematicians who still believe strongly in "the old time religion" and cannot see what this fuss is all about.

In view of this complex of inertial elements, it is the opinion of this Committee that, in mathematics, a broad, coordinated attack will be needed in contradistinction to the textbook revolution which was successful in the teaching of English literature. In elementary mathematics, new ideas introduced by local writing and teaching efforts have always turned out to be largely overpowered by the self-propagation of the traditional but retrogressive stock. Moreover, we believe that for very cogent reasons we must continue to seek *one, universal freshman course* for all reasonably qualified students, presupposing intermediate algebra and high school geometry, and ignoring the question of remedial courses for students who enter college without adequate mathematical training in school. Only by means of such a universal course can the best principles of liberal education be served. Only in this way can we avoid the error of forcing the immature student, upon entering college, to make choices which will seriously restrict his freedom of development in later years, and in this way we will attempt to make sure that the subjects taught are really the most valuable ones. We are confident that the mathematics which survives such a selection process from the common

intersection of all of demands of the various disciplines, including higher mathematics, will turn out to have the values of generality and abstractness which have always characterized good mathematics. Also we are confident that this selection process will naturally sift out essentially the same set of ideas which would be chosen as fundamental if one considered only the development of mathematics itself.

**2. A universal freshman course and sequels.** The proposed universal course would contain all of the theory prerequisite to any sophomore course in mathematics, covering this, with appropriate exercises and problem solving, in three hours per week for a year. In most schools of engineering, it should be supplemented by technical problem-solving work in additional "laboratory" class meetings.

To assure ourselves of the existence of a mathematics course for freshmen which meets these specifications, the Committee has attempted to construct one in outline. We do not propose our course as the solution but perhaps a schematic diagram might illustrate the general direction of our efforts. The "technical




laboratory supplement," which might be two hours per week, would presuppose a stronger high school training, perhaps three units, and this strong high school training, jointly with the universal freshman course, would form the prerequisite for the sophomore calculus. The laboratory itself would not be a prerequisite to sophomore calculus, though it could serve a remedial function which might fit many weakly prepared students. These supporting functions of the technical laboratory work are indicated by the dotted arrow pointing towards calculus. This scheme would continue to make essential use of strong mathematical training in high school without shutting the door to a good student who enters college without such training.

This Committee thought of the universal course as being made up of two subjects. (Arrange them as you will.) One subject would consist of functions, graphs and elementary calculus, covering differentiation and integration of polynomials, logarithms, exponential functions and general powers, with a minimum of formulas and with emphasis on ideas. The other subject would take off from set concepts, set up the fundamental language of theoretical mathematics and proceed, via some of the simpler abstract algebraic systems (but not the full real number system), to probability with emphasis upon the binomial distribution. If possible, this should lead to the sampling problem and here make a junction with statistics without actually getting into statistics. We believe that the universal course should contain little or no statistics but it should lay the foundation for this subject in a much more thorough way than the traditional course does. Engineers and physical scientists now need these probability ideas as well as the social scientists.

Now in this scheme, what becomes of the traditional college algebra, trigonometry, and analytic geometry? The essential algebra in college algebra would be covered in the universal course. Numerical trigonometry might be handled in the technical supplement. Analytic trigonometry might form the first subject of the sophomore calculus, which would be time enough for its uses in mechanics. Euclidean analytic geometry is well known to be not a proper prerequisite to calculus, which requires the more general geometry, the geometry of the cartesian product of the real line and the real line, *i.e.*, "graphs." This includes straight lines, slopes and the (weak) area of a rectangle. Thus the first real treatment of *Euclidean* analytic geometry would be in the sophomore course. (This beautiful old subject needs a proper revival in American college mathematics in a way which neither the hurried freshman nor sophomore courses permit!) Moreover, the graphs, or affine geometry, of the Universal Course, is not a negligible geometry.

The schematic diagram indicated a branching in the second year with a course called mathematics for social sciences and statistics appearing parallel to the traditional calculus. This is a long-needed development. For the calculus has not been a suitable universal second course. Actually, however, this alternate second course will probably turn out to meet many needs other than those of the social science student and potential statistician. For example, the prospective high school teacher might find it better for his needs than calculus. What the syllabus of such a new course should contain is an open question. Statistics? Multivariable algebra? Postulational models? Scales of measurement? Maximum problems in several variables?

Returning to the technical laboratory of the first year, we think of the first semester being devoted to numerical trigonometry and to numerical methods, with considerable practice in graphical methods and the use of tables of logarithms and exponentials. Worded problems would form an important part of it, with calculus drill beginning after mid-semester. Since relatively little calculus

would appear in the second semester of the universal course, the technical laboratory for the second semester would become largely a calculus drill course. Thus engineering students would enter the sophomore course in analytic geometry and calculus with almost a year's study of elementary calculus behind them, a good part of which would have been based upon a strong high school training.

By these means and by providing at minimum cost an appropriate outlet for those who want and need less technical mathematics, we hope to strengthen the foundations of the classical sophomore calculus course and enable it to be started at a relatively advanced point and to proceed into the methods and applications of calculus. This should clear the way for more vector methods, at least in the second semester, and perhaps permit the calculus to include more differential equations, mechanics, and numerical integration than has been possible in the past. Finally, we point out that the principle of the Universal Course provides for an efficient conversion of those students who discover, after entering college, that they want to study engineering or physical science. The so-called "terminal course" often makes this change of plan prohibitive in cost.

The Committee has nothing of its own to report on courses beyond the sophomore year. It has been seeking out interesting new courses which are being offered in various institutions, some of which are mentioned in Section 3 of this report. A number of the discussions of the Committee have centered on the revival of geometry.

**3. Summary of known activities.** For the answers to the questions about the proposed sophomore course in mathematics for social scientists, and other questions on mathematics basic to social science, our Committee is relying upon another group, *viz.*, the Committee on the Mathematical Training of Social Scientists which developed into the 1953 Summer Institute on Mathematics for Social Scientists at Dartmouth, Professor William G. Madow, Chairman. This very significant project was described at the Association Christmas Meeting in Baltimore by several members of the Institute: W. G. Madow, R. M. Thrall, R. R. Bush, and Howard Raiffa.* In addition to this group, K. O. May of Carleton College is working on an interesting book on mathematics for social scientists which has much new material. Also W. G. Madow is individually writing an elementary book on mathematics for social scientists. This is significant in view of the author's experience in the Dartmouth Summer Institute. Another body of material which might go into the Mathematics for Social Science is the semester course in statistics as constructed by S. S. Wilks for Princeton freshmen to follow a semester of calculus.

Since there existed no such Committee on the important questions relating to the mathematical training for the engineering of today and of the future, a joint committee was appointed by the Association and the American Society for

* See this MONTHLY, vol. 61, pp. 550–561.

Engineering Education. Professor G. B. Thomas of M.I.T. is Chairman. This Committee will search out the engineering leaders who have the knowledge and foresight to tell us what forms of mathematical training are likely to be most valuable in the engineering of the next twenty years. To assist this committee in its work it is hoped that some conferences on modern mathematics and modern engineering will be arranged, possibly with the sponsorship of the National Science Foundation. R. S. Burington, who is a member of the joint committee, and also president of the Mathematics Section of A.S.E.E., arranged an extremely interesting program on this subject at the 1954 summer meeting of A.S.E.E. in Urbana, Illinois. It is hoped that the proceedings will soon be available in print.*

At a somewhat higher level, training in applied mathematics has been the subject of a study directed by F. J. Weyl for the Division of Mathematics of the National Research Council. Two symposia were held in connection with meetings of the Society. Proceedings of these symposia will be available in print soon.** This work was supported by the National Science Foundation.

The idea of early introduction of calculus, inherent in the proposed Universal Course, is by no means new or radical. This is the form of the first college course implied by both of the studies of articulation of school and college supported by the Fund for the Advancement of Education. The first report appeared in *General Education in School and College*, Harvard University Press (1952). The mathematical part was reprinted in AMERICAN MATHEMATICAL MONTHLY.§ The mathematical part of the second study is known as the Brinkmann Committee, whose results were reported to the Association at the Baltimore meeting, December 1953.§§ The older textbooks which form prototypes for at least the first half of the Universal Course include those of F. L. Griffin, *Mathematical Analysis;* and Milne-Davis, *Introductory College Mathematics*, the latter being interesting also for its sections on numerical methods. The use of such books with an early introduction to calculus dispensing with the traditional "preparation" is increasing and meeting with good success, notably in the California schools, and in Minnesota, following the University of Minnesota. Other recent books on freshman calculus of special interest are those of D. E. Richmond of Williams College, E. G. Begle of Yale, and Karl Menger of Illinois Institute of Technology. Begle's book has for this level perhaps the most modern treatment of the formal theory available in print. Menger's book contains a radical attack on the conceptual ideas of the variable and proposes a solution using the identity function as a variable. This book may have great influence, for notable simplifications are achieved.

A unique text in many ways is the preliminary edition of Allendoerfer-Oakley

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* See this MONTHLY, vol. 62, 1955, pp. 385–392.

** See this MONTHLY, vol. 61, No. 7, Part II, Slaught Memorial Paper, No. 3.

§ See this MONTHLY, vol. 60, 1953, pp. 380–383.

§§ See this MONTHLY, vol. 61, 1954, pp. 319–323.

*Principles of Mathematics*.* This text contains the best material on sets, logic, and Boolean algebra of any informal general mathematics text.

A number of books have been written or are in writing which seek to introduce mathematics in a more formal and logical manner than is conventional. These include *Fundamental Mathematics*, written by the College Mathematics Staff of the University of Chicago. It aims at an organization and explication of basic mathematical concepts, built up from elements of logic and set theory. Also at Chicago, the Department of Mathematics is preparing texts for undergraduate courses. In some respects Herman Meyer of the University of Miami, in his *Mathematical Analysis*, pushes the postulational development of the traditional freshman material farther and more consistently than any other text. Also in this field, the Johnson, McCoy and O'Neill *Fundamentals of College Mathematics* has been used experimentally in a number of places.

At the University of Kansas this summer, the Kansas Summer Writing Group under the direction of G. B. Price is writing a preliminary edition of text material for the Universal Course as designed by the Committee. This text material will be tried out on a full scale at Tulane and on a smaller scale at several other institutions next year. This non-commercial writing was supported by the University of Kansas and the Social Science Research Council.

Courses which show interesting developments, and of which we are aware, include Artin's honors course for specially selected freshmen at Princeton. This goes further in calculus than conventional courses do in two years. The Stanford University department is constructing an experimental honors course in calculus for social scientists, having a course in logic taught in the philosophy department as a prerequisite. Most mathematicians are familiar with Pólya's work, also at Stanford, on heuristic methods in mathematics. His forthcoming book on *Mathematics and Plausible Reasoning*** is awaited with much interest. New courses in appreciation and understanding of mathematics are being offered, but the returns are not yet in on the support for them. Besides courses based upon the books of R. L. Wilder, E. R. Stabler, Courant-Robbins, and Kershner-Wilcox, we know that W. Feller at Princeton and A. M. Gleason at Harvard are developing courses of this character. At Brown University, H. Federer, Jonsson, and K. G. Lister have been developing a four-semester course in elementary mathematical analysis from a modern point of view, based on text material written by Federer and Jonsson. A significant feature is the exercises for students in mathematical writing, with complete quantification and logical detail. Among the more advanced undergraduate courses, A. W. Tucker has a well developed undergraduate course in combinatorial topology and R. L. Bing at Wisconsin has one on elementary set topology.

New courses and programs for training teachers of mathematics, either for high school or college teaching, are too numerous to mention in detail here,

* Printed edition, McGraw-Hill Book Company, 1955.

** Princeton University Press, 1954.

especially since they are in the province of the Newsom Joint Committee of M.A.A. and N.C.T.M. Briefly we may indicate the internship program for prospective college teachers, sponsored by the Fund for the Advancement of Education at nine universities, and new graduate programs, such as that at Yale. Also the special emphases of the University of Wisconsin and Notre Dame departments in this direction are noteworthy. The first Summer Conference for College Mathematics Teachers last year at Boulder was organized by B. W. Jones. New ones are being directed this summer by Ivan Niven at Oregon, E. A. Cameron at North Carolina, and for high school teachers by C. B. Allendoerfer in Seattle, all sponsored by the National Science Foundation. A college lectureship program for the coming year is being set up with a grant from N.S.F. The Internationale Matematische Unterrichts Kommission, founded by Felix Klein, is being revived at the Amsterdam International Congress. Saunders MacLane, S. S. Cairns, and A. M. Gleason represent the United States in that activity.

**4. An appeal to all mathematicians.** We here appeal to mathematicians, teachers, scientists, and engineers to give us the benefit of your counsel. Tell us what you are doing and what you think is needed. Tell us also specifically what this "dead wood" is which everyone says we can cut out of the present program. Give us copies of your notes or syllabi for new courses and tell us about your experiments in teaching or writing. We are particularly concerned to have the results of self-study analyses which departments of mathematics have conducted. Ultimately any complete job of writing must be in the form of writing for the classroom text with exercises. Anything short of that is likely to be of very transient value. If possible, send us seven copies of any such material you have which you consider to be significant for a reformulation of undergraduate mathematics, especially now the freshman program, but ultimately reaching on to the advanced work and back into the schools. This information will be directly helpful in carrying out the work itself and indirectly it will provide us with a basis upon which we can assess the amount of financial assistance which will be needed in furthering the program on a national basis.

Direct approaches to the physiologists, physicists, and earth scientists have been initiated with a view to getting counsel from them, analogous to that which we sought from the engineers and social scientists. Scientists in other fields will be approached later, and the results will be presented in another report.

Meanwhile, in mathematics itself there is the most urgent need for the active participation of the very best living mathematicians in the work of reformulation of our school and college programs. The last successful reformulation of the program in mathematics was accomplished in the early part of this century and was founded upon the ideas of such outstanding mathematicians and teachers as Felix Klein, Poincaré, Boltzmann, E. H. Moore, J. W. A. Young, John Perry, and many others. That a number of subsequent reform efforts have been failures or at most minor successes may be attributed to their lack of mathematical

leadership; for the teaching organizations tried to go it alone. This Committee is seeking to enlist the active participation of our able mathematicians in this work but, independently of our efforts, many established and young mathematicians who formerly thought primarily in terms of research are turning their attention to the teaching of mathematics. For it is realized that the economic basis of the mathematical profession and its relation to the general welfare is still largely founded on teaching. If graduate departments of mathematics do not furnish the teachers of college mathematics, and these teachers are qualified in some other way, we may expect that the values of modern mathematics will be lost to all but a few specialists. Moreover, the economic basis of the profession will be destroyed, as it was in the profession of classics. Unfortunately, many of the leaders of the mathematical profession are in universities where they are insulated from the signs of deterioration which are apparent in the average college and hence do not have much direct experience with the problem.

**5. Suggested Special Undertakings.** Finally, we list a few possible subjects for special projects which will contribute to the general objectives. We hope that individual mathematicians and departments will undertake them. If financial assistance is needed, this Committee will undertake to be of such assistance as it can to those who plan to submit a proposal asking for it. The necessary coordination of these financial appeals is the business of a committee under the chairmanship of Dean Mina Rees of Hunter College. A partial list of special problems on which work is needed is: (1) The writing of several manuscripts for a universal freshman course and classroom trials of these text materials. (2) The simplification of calculus so that it can be taught to freshmen generally. (3) Mathematics for genetics and physiology. (4) Mathematics for the earth sciences. (5) Theory for first year students. For example, how far should one go into logic? How well do rigorous postulational treatments get across? (6) Correct and teachable set theory and probability ideas for the first year. (7) The role, if any, of the normal probability distribution. (8) Dimensional analysis and measurement. (9) Vectors and linear operators as early as possible and in calculus. (10) Heuristic methods and problem solving to teach students to *make* the abstraction rather than to teach them previously abstracted theory. (11) The resetting of geometry in the curriculum. (12) Calculus technique making simplest use of numerical methods, galleries of graphs and tables of integrals. (13) Assembling and preparing readings, other supplementary materials for use in elementary courses, and guides which will make the mathematics library more accessible to students. (Here H. L. Meyer and I. Wirszup of the Chicago College Staff are doing good work.) (14) Undergraduate "research." (15) New courses in pure mathematics for undergraduates (such as R. L. Bing's topology course). (16) Combinatorial analysis and digital theory. (17) Boolean algebra. (18) Courses in computation. (19) Number theory for undergraduates and perhaps upper high school students. (20) Mathematics in the arts. (21) Complex numbers and functions in the undergraduate curriculum. (22) Trigonometry. What

is its place and value? (23) Grassman algebras and exterior products. Shall the 3-dimensional vector product be retained in courses? (24) School-college articulation effecting the advancement of well-trained students. With college credits? (25) Training for applied mathematics. (26) Morale building in high school mathematics. (27) Training for actuarial science. (28) Tukey's "mathematical engineering." (29) Reformulation of standard undergraduate programs.

**6. Publication Policy.** In conclusion, we state our policy on copyrights and relations to commercial publishers. This Committee will not get into the textbook business and will not approve or disapprove any textbook. Writers of experimental text materials, when subsidized by grants under the jurisdiction of this Committee will assign all royalty interests to the Association or the foundation making the grant until the expiration of the period of the grant is reached or the repayment of the grant has been effected. Ultimately we expect that any new textbooks, which may result wholly or in part from our efforts, will be published by normal methods. We must treat whatever ideas come to us as belonging to the public domain, though we hope we will always acknowledge credit where it is due. With these understandings we invite every mathematician to contribute his own ideas to the general effort.

Along with the undersigned members of the Committee, President E. J. McShane has worked as hard as any member of it, and R. C. Yates contributed greatly as an active member in 1953.

August 16, 1954

W. L. DUREN, JR., *Chairman*
C. V. NEWSOM
G. B. PRICE
A. L. PUTNAM
A. W. TUCKER

**ADDENDUM**

The Committee on the Undergraduate Program is being reorganized with a new Chairman, E. J. McShane, who originally appointed it. The new Committee will also include John G. Kemeny of Dartmouth. W. L. Duren will remain an ex officio member as President of the Association.

The grant of $25,000 made in March 1955 by the Ford Foundation, and announced elsewhere in this MONTHLY, gives the Committee its first dependable financial support. This amount of money is generous for a start but the magnitude of the general effort is so great that these funds are small in comparison.

The Committee met in Lawrence in January 1955 with further support from the University of Kansas. Here a week's work was done on Universal Mathematics, Part II, particularly on applications of the Boolean set operations. A new Summer Writing Group is now being planned for 1956. Its location has not been determined. A critical account of the Tulane experience with Universal Mathematics, Part I, has been prepared for publication in this MONTHLY.

April, 1955

W. L. DUREN, JR.

## NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 124 persons have been elected to membership by the Board of Governors on applications duly certified.

GRACE L. ABHAU, Student, Kent State University.

EUGENE ALBERT, M.A. (Brooklyn C.) Instr., Brooklyn College.

HARMON ALEXANDER, M.A. (N.Y.U.) Van Nuys, Calif.

W. O. ALEXANDER, JR., B.S. (Houston) Grad. Student, University of Houston.

ELIZABETH A. AULT, Student, Ursinus College.

A. F. BAKENHUS, Student, University of Houston.

LEONARD BANNER, B.A. (Brooklyn C.) Math. Teacher, Lubavitcher High School, Brooklyn, N. Y.

G. E. BILLS, A.B. (W. Va. U.) Lt. (j.g.) United States Navy.

L. M. BOSTICK, B.S. (Eastern New Mexico) Grad. Asst., Eastern New Mexico University.

R. D. BOYD, Student, Queen's University.

J. S. BRADLEY, Student, Mississippi Southern College.

W. S. BUSH, B.A. (Mississippi) Grad. Fellow, University of Mississippi.

RAYMOND CARPENTER, M.A. (Columbia) Asso. Professor, Northeastern State College, Tahlequah, Okla.

J. R. CASTILLO-NUNEZ, Student, Agricultural and Mechanical College of Texas.

W. E. CHRISTILLES, Student, St. Mary's University.

LEROY COOPER, B.S. (Philander Smith) Tulsa, Okla.

J. B. DOUGLASS, Student, Texas Christian University.

REV. E. D. EICHMAN, Math. Teacher, Epiphany Apostolic College.

A. H. ETTELSON, Student, University of Buffalo.

D. L. FAASS, Student, University of Houston.

HERBERT FARKAS, Student, City College of the City of New York.

F. D. FARLEY, B.S. (Concord) Grad. Student, West Virginia University.

F. G. FENDER, Ph.D. (Pennsylvania) Professor, Rutgers University.

A. M. FENSTERMACHER, Student, Knox College.

LEAH FINE, Student, Agnes Scott College.

P. M. FITZPATRICK, M.S. (Catholic) Pvt., United States Army.

J. L. C. FORD, JR., Student, Montana State College.

T. S. FRANK, Student, Lawrence College.

R. G. FRYER, B.S. (St. Lawrence) Teaching Fellow, University of Buffalo.

A. V. GAFARIAN, B.S.E. (Michigan) Grad. Student, University of California at Los Angeles.

D. T. GIANTURCO, Student, University of Buffalo.

M. C. GILLILAND, B.S. (U.C.L.A.) Grad. Student, University of California at Los Angeles.

HARRISON GIVENS, JR., B.S. (Yale) Equitable Life Assurance Society, New York, N. Y.

JUDITH GORENSTEIN, Student, Massachusetts Institute of Technology.

L. A. GRAHAM, A.M., M.E. (Columbia) Engr., Graham Transmissions, Menomonee Falls, Wisc.

E. A. GREEN, Student, Queen's University.

MRS. ELZIE A. GREENE, M.A. (Arkansas) Staff Member, Sandia Corp., Albuquerque, N. M.

KATHLEEN HAMLIN, Student, Wayne University.

J. V. HANCOCK, Student, Memphis State College.

W. J. HARTMAN, Student, University of Colorado.

LT. COL. J. C. HEMPSTEAD, C. E. (Iowa S.C.) Asst. Professor, United States Air Force Academy.

W. F. HERRIN, Student, Central College.

T. W. HILDEBRANDT, S.M. (M.I.T.) Grad. Fellow, Oak Ridge Institute of Nuclear Studies.

W. A. HOCKINGS, B.S. in Chem. (Michigan C. of Mining and Tech.) Res. Chemist, Calumet & Hecla, Calumet, Mich.

R. F. HOUSE, Student, Texas Christian University.

H. C. HOWARD, JR., B.A. (Wooster) Teaching

Asst., Carnegie Institute of Technology.

LORETTA A. HUMPHREYS, Student, University of Oregon.

G. R. HUNT, M.A. in Ed. (New Mexico) Asst. Professor, San Angelo College.

M. J. HURLEY, JR., Student, University of Arizona.

RONALD JACOBOWITZ, Student, City College of the City of New York.

J. D. JACOBS, Sycamore, Va.

E. S. KEEPING, D.I.C. (London) Professor and Head of the Department of Mathematics, University of Alberta.

C. H. KEIM, Midshipman, United States Naval Academy.

D. F. KIMBALL, Student, Brown University.

D. R. KING, Student, Rutgers University.

T. C. KIPPS, M.A. (California) Instr., University of Santa Clara.

H. P. KUANG, M.S. (Minnesota) Teaching Asst., University of Minnesota.

L. J. LANGE, B.S. (Regis) Grad. Student and Instr., University of Colorado.

R. A. LEACH, B.A. (Roosevelt) Engr., Natural Gas Pipeline Company of America, Chicago, Ill.

JOANN LETOHA, Student, Kent State University.

W. C. LORDAN, B.A. (Wesleyan U.) Grad. Student, University of Wisconsin.

J. H. P. MAECHER, M.S. (Miami) United States Army.

R. T. J. MAHONEY, Student, University of Buffalo.

D. B. MAIER, Student, Rutgers University.

R. L. MANCHESTER, B.A. (Buffalo) Math. Instr., Franklinville Central School, N. Y.

M. J. MANSFIELD, M.S. (Purdue) Res. Asst., Statistical Lab., Purdue University.

SAMUEL MAREIN-EFRON, Student, Harvard University.

A. V. MARTIN, Ph.D. (Duke) Asso. Professor, University of New Mexico.

T. D. MCADAM, A.B. (Washburn) Instr., Washburn University of Topeka.

C. A. MCCARTHY, Student, University of Rochester.

BROTHER ANDREW MCCAULEY, B.A. (Manhattan) Grad. Student, Catholic University; Instr., De La Salle College.

L. L. MCCELVEY, B.A. (A.&M.C. of Texas) 2nd Lt., Pilot, United States Air Force.

J. B. MCGRORY, Student, University of the South.

G. W. MCLAUGHLIN, B.A. (Washington S.C.) Private Tutor, Harrington, Me.

J. P. MENARD, B.A. (St. Michael's) Grad. Teaching Asst., Catholic University.

D. M. MERRIELL, Ph.D. (Chicago) Head, Department of Mathematics, Robert College, Istanbul, Turkey.

AARON MILLER, M.S. (Washington U.) Retired Math. Teacher, Indianapolis, Ind.

W. G. MILLER, Ph.D. (Florida) Professor, Clemson College.

MISS MERLE MITCHELL, M.A. (New Mexico) Instr., University of New Mexico.

J. L. NEWTON, Student, University of Colorado

R. V. NOLAN, B.A. (Buffalo) Teaching Fellow, University of Buffalo.

R. E. OBERDORFER, Student, Kent State University.

A. J. O'CONNOR, Schenectady Sign Service, N. Y.

T. J. O'NEIL, B.A. (Buffalo) Grad. Student, University of Buffalo.

R. C. O'NEILL, M.A. (Columbia) Lt. (j.g.) U.S.N.R.

MARY F. OVERFIELD, Student, Kansas State Teachers College.

J. H. PADGETT, M.A. (North Carolina) Instr., Armstrong College.

HIRAM PALEY, Student, University of Rochester.

G. L. PATE, M.A. (George Peabody) Instr., University of Georgia, Atlanta Division.

G. P. PATERNOSTER, Structural Detailer, Harry L. Dovell & Co., Chicago, Ill.; Student, Roosevelt University.

SIDNEY PENNER, Student, City College of the City of New York.

R. L. PRATT, Student, Washington University.

SUSAN PYEATT, Student, Catholic University.

J. T. ROBERTSON, Student, Memphis State College.

J. B. ROGERS, M.A. (Michigan) Math., Rand Corporation, Santa Monica, Calif.

HUGO ROSSI, Student, City College of the City of New York.

G. L. ROWLAND, A.B. (U.C.L.A.) Instr., University of New Mexico.

RUTH L. ROYER, M.S. (Iowa S.C.) Instr., Chico State College.

T. W. ROZELLE, M.A. (Michigan) Asso. Professor, Wisconsin State College.

A. A. SAGLE, Student, University of Washington.

R. W. SCOTT, B.A. (Carleton) Northfield, Minn.

C. R. SELIGER, A.B. (Johns Hopkins) Teaching Asst., Rutgers University.

R. E. SHAFER, Student, University of California, Berkeley.

O. T. SHANNON, M.Litt. (Pittsburgh) Asst. Professor, Agricultural, Mechanical & Normal College, Pine Bluff, Ark.

L. A. SHEPP, Student, Polytechnic Institute of Brooklyn.

A. J. SILBERGER, Student, University of Rochester.

M. G. SMITH, B.S. (Oklahoma) Teaching Asst., University of Oklahoma.

DANIEL SOKOLOWSKY, M.S. (Wisconsin) Asst. Professor, Antioch College.

S. E. SPIELBERG, Student, University of Pennsylvania.

J. C. STUELPNAGEL, Student, Yankton College.

REV. T. J. TAYLOR, M.S. (Notre Dame) Instr., St. Ambrose College.

GEORGIA M. THOMPSON, Student, University of Arizona.

GABRIEL TSIANG, M.S. (Notre Dame) Lecturer, Southern Illinois University.

J. B. VIEAUX, Student, Agricultural and Mechanical College of Texas.

T. C. WALKER, Student, Montana State University.

H. M. WEITKAMP, M.A. (Cincinnati) Teacher, Cincinnati Public Schools.

L. W. WESLEY, B.A. (Minnesota) Supervisor, Engg. Dept., North American Aviation, Columbus, Ohio.

ROBERT E. WHEELER, Airman, United States Air Force.

FREDERICK R. WHITE, Student, University of Buffalo.

K. F. WILSON, B.S. (Florida) Grad. Asst., University of Florida.

R. J. WILSON, Student, St. Mary's University.

T. J. WOLINSKI, Student, St. John's College.

G. F. WOODLIEF, JR., Student, Duke University.

J. L. WULFF, A.B. (Sacramento S.C.) Grad. Student, Sacramento State College.

## THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 26, 1955, at Michigan State College in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor R. V. Churchill of the University of Michigan presided at both morning and afternoon meetings and at the luncheon and business meetings.

A total of sixty-one persons attended the meetings including forty-four members of the Association:

Bess E. Allen, W. D. Baten, F. A. Beeler, J. E. Bellardo, C. H. Butler, A. T. Butson, R. V. Churchill, S. D. Conte, Helen E. Core, J. W. Coy, D. E. Deal, E. R. Deal, S. F. Dice, H. G. Falahee, J. S. Frame, J. W. Gaddum, E. W. Goings, V. G. Grove, L. W. Gunter, Frank Harary, G. E. Hay, Fritz Herzog, J. D. Hill, Walter Hoffman, R. D. James, Leo Katz, M. A. Laframboise, G. P. Loweke, M. T. MacNeil, L. E. Mehlenbacher, D. C. Morrow, H. W. Nace, A. L. Nelson, E. A. Nordhaus, E. S. Northam, George Piranian, J. E. Powell, J. H. Powell, E. D. Rainville, R. W. Schenkel, F. C. Sherburne, Jr., B. M. Stewart, C. K. Tsao, J. L. Ullman.

The nominating committee consisting of Professors B. M. Stewart, Chairman, and D. C. Morrow proposed Professor C. C. Richtmeyer of Central Michigan College of Education as Chairman and Professor S. D. Conte of Wayne University as Secretary-Treasurer for the year 1955–1956. The slate was elected unanimously.

Professor C. H. Butler was appointed chairman of a committee of the

Michigan Section of the MAA which is to cooperate with a similar committee from the Michigan Council of Teachers of Mathematics with the purpose of improving the quality of mathematical teaching in the secondary schools and of encouraging students to continue their mathematical training.

The following papers were presented at the morning and afternoon sessions:

1. *Some properties of n-convex functions*, by Professor R. D. James, Michigan State College.

A convex function $f(x)$ is one for which the curve $y=f(x)$ between $x_1$ and $x_2$ lies below the chord joining the points $(x_1, y_1)$, $(x_2, y_2)$. It is well known [Hardy, Littlewood, and Pólya, *Inequalities*, Theorem 111] that a continuous convex function has left-hand and right-hand derivatives. A convex function may also be defined as one whose divided differences of order 2 are all non-negative, and this concept may be generalized. A function whose divided differences of order $n$ are all non-negative will be called an $n$-convex function. With this definition a 2-convex function is convex in the ordinary sense. It can be shown that an $n$-convex function has differential coefficients of order $r$, $1 \leqq r \leqq n-2$ and left-hand and right-hand differential coefficients of order $n-1$.

2. *An application of the Cauchy-Hadamard formula*, by Professor J. L. Ullman, University of Michigan.

In a first course in function theory the Cauchy-Hadamard formula for finding the radius of convergence of a power series is introduced, and it is pointed out that a singularity of the function defined by the series must lie on the circle of convergence. Our purpose is to show how the Cauchy-Hadamard formula can be used for arriving at deeper properties of functions. The results discussed are a) a method for finding all the poles of a meromorphic function, b) the location of the limit points of the totality of the zeros of a meromorphic function and its derivatives, and c) a proof of a theorem due to Jentzsch, together with some generalizations.

The exposition is based on the following references: a) A. Hurwitz, *Sur une Theorème de M. Hadamard*, Comptes rendus des séances de l'Academie des Sciences, vol. 128, 1899, pp. 350–353; b) George Pólya, *Über die nullstellen sukzessiver derivierten*, Mathematische Zeitschrift, vol. 12, 1922, pp. 36–60; c) J. L. Ullman, *Hankel determinants whose elements are sections of a Taylor series*, Part I, Duke Mathematical Journal, vol. 18, 1951, pp. 151–156, Part II, same journal, vol. 19, 1952, pp. 155–164.

3. *On complementary graphs*, by Professor E. A. Nordhaus and Dr. J. W. Gaddum, Michigan State College, presented by Professor Nordhaus.

Best possible bounds are established for the sum and product of the chromatic numbers of an unoriented finite linear graph and its complementary graph. Estimates are also given for upper and lower bounds for the chromatic number for graphs having a prescribed number of nodes and edges, employing a result of Turan. Some results are found concerning the density of graphs by correlating recent work done by Zykov, Greenwood and Gleason.

4. *On strengthening and weakening points of directed graphs*, by Professor Frank Harary, University of Michigan.

Four degrees of connectedness: strong, unilateral, weak, and disconnected, are defined for directed graphs (digraphs). These satisfy: every strong digraph is unilateral; every unilateral digraph is weak. Let $C_3 = \{\text{strong digraphs}\}$, $C_2 = \{\text{unilateral digraphs which are not strong}\}$, $C_1 = \{\text{weak digraphs which are not unilateral}\}$, $C_0 = \{\text{disconnected digraphs}\}$ be the disjoint connectedness categories. Let $\Pi_{ij}$ be the set of all points $P$ such that digraph $\mathcal{D} \in C_i$ and $\mathcal{D} - P \in C_j$. A point $P \in \Pi_{ij}$ is called a strengthening point of $\mathcal{D}$ if $i > j$, and a weakening point if $i < j$. It is shown that $\Pi_{31}$ is the only empty class among the $\Pi_{ij}$. Characterizations of other classes $\Pi_{ij}$ are given and strengthening lines are also discussed.

5. *Stability of numerical methods in the solution of differential equations* by Professor Saul Rosen, Wayne University. Presented by title.

6. *An engineering degree with specialization in mathematics*, by Professor G. E. Hay, University of Michigan.

An expository paper dealing with the mathematical content of an engineering degree which emphasizes mathematics. This degree is currently offered at the University of Michigan.

7. *A note on power series which diverge everywhere on the unit circle*, by Professor Fritz Herzog, Michigan State College.

Lusin has shown that there exist power series $\Sigma a_n z^n$, with $\lim a_n = 0$, which diverge at all points of the circle $|z| = 1$. In this note, a very simple example of a power series of this kind is constructed, which has the additional property that its coefficients are real and non-negative. Moreover, the example can be modified in such a way that the real part of the power series yields an example of a cosine series $\Sigma a_n \cos n\theta$, with $a_n \leqq 0$ and $\lim a_n = 0$, which diverges for all real $\theta$.

8. *On a certain family of subsets of a set*, by Professor Ben Dushnik, University of Michigan, introduced by the Secretary.

A condition is stated under which a set $\epsilon$ can be represented as the union of a specified number of members of a certain family of subsets of $\epsilon$. The condition is related to the generalized continuum hypothesis.

9. *Some properties of the Fejér polynomials*, by Professor Fritz Herzog, Michigan State College, and Professor George Piranian, University of Michigan, presented by Professor Piranian.

The polynomials

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \frac{z^{n+1}}{2} - \cdots - \frac{z^{2n-1}}{n}, \qquad n = 1, 2, \cdots,$$

were introduced by Fejér, who showed that they are uniformly bounded on the unit circle. The authors show that, as $n \to \infty$,

$$\lim M_n = 2 \int_0^\pi \frac{\sin t}{t}\, dt,$$

where $M_n$ is the maximum of $|P_n(z)|$ on the unit circle. They also obtain the following results on the location of the roots of $P_n(z) = 0$. The point $z = 1$ is the only positive root and the only root on the unit circle. Each of the sectors $(2k-1)\pi/n < \arg z < (2k+1)\pi/n$, $k = 1, 2, \cdots, n-1$, contains exactly two roots. There are, therefore, no negative roots when $n$ is odd. But there are two negative roots when $n$ is even, and their distance from the point $z = -1$ is asymptotically equal to $(\log n)/n$, as $n \to \infty$. Finally, all roots, other than $z = 1$, lie at a distance from the unit circle which is less than $(4+\epsilon)(\log n)/n$ and greater than $[(2e)^{1/2} - \epsilon]/n$, if $n$ is sufficiently large.

10. *The abstract definition of conditional probability and conditional expectation*, by Dr. Shu-Teh C. Moy, Wayne University, introduced by the Secretary.

The definition of the conditional expectation of a random variable, or the conditional probability of an event with respect to another random variable, is based on the Radon-Nikodyn theorem of derivatives. For random variables which take values in a separable, reflexive Banach space, a similar theorem of derivatives is proved. Based on this theorem, the conditional expectation of a Bochner integrable function can be similarly defined.

11. *A continued fraction for periodic rent, logarithms, and roots*, by Professor J. S. Frame, Michigan State College.

The continued fraction defined recursively by setting $f_k(n, r)=f_k=(n^2-k^2)r^2/((2k+1)(2+r)+f_{k+1})$, is rapidly convergent if $x=nr<1$ and $r<1$, and it converges for all $r>-1$ when $n=0$. It provides a rapidly convergent expression for the total repayment $T_{n,r}=n/a_{\overline{n}|r}=1+(n+1)r/2+f_1(n, r)/2$, on a unit loan repaid in $n$ equal installments with compound interest at rate $r$ per period. In fact, if $x=nr<1$ and $r<1/2$ the fourth convergent has an error of about $2(x/10)^{10}$ and gives $T_{n,r}$ to ten significant figures. Letting $n$ approach 0 yields $\ln(1+r)=r/T_{0,r}$, convergent for all $r>-1$. Setting $n=x/r$, and letting $r$ approach 0, gives the familiar continued fraction $f_1=x\coth(x/2)-2$ where $f_k=x^2/(4k+2+f_{k+1})$. Finally a rapidly convergent root extraction method is defined by $(c^m+r)^{1/m}=c+2cr/(mb-r-F_1)$ where $b=2c^m+r$ and $F_k=(k^2m^2-1)r^2/((2k+1)mb-F_{k+1})$.

12. *The Jacobi integral in the restricted problem of three bodies applied to elliptic orbits*, by Professor G. P. Loweke, Wayne University.

The Jacobi integral in the restricted problem of three bodies can be applied to elliptic orbits of the finite masses at their maximum and minimum distances by observing that at these extreme positions the orbits of the two masses reduce to circles. At minimum distance corresponding to perihelion the angular acceleration is zero, changing from positive to negative at this point, and the motion is perpendicular to the radius vector. At a point corresponding to aphelion this condition again exists, the angular acceleration changing from negative to positive at this point. The surfaces of zero relative velocity for elliptic orbits can now be compared at these two points. It can be shown that the error involved by assuming a circular solution is a function of the eccentricity of the orbits of the finite masses, $e^2$. When applied to the major planets the maximum per cent error by disregarding the eccentricity is as follows: For the orbit of Mercury, 75%; Venus, 2%; Earth, 5%; Mars, 30%; Jupiter, 15%; Saturn, 18%; Uranus, 15%; Neptune, 3%; Pluto, 95%.

S. D. CONTE, *Secretary*

## THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 11–12, 1955, at Tennessee Polytechnic Institute, Cookeville, Tennessee. Professor W. L. Williams, Chairman of the Section, and Professor R. H. Moorman, Vice-Chairman, presided over the general sessions; Professors R. W. Cowan, L. A. Dye, Tomlinson Fort, C. G. Phipps, E. B. Shanks and W. S. Snyder presided over subsections.

There were about 180 in attendance including the following 99 members of the Association:

R. H. Ackerson, S. A. Anderson, R. W. Ball, R. C. Boles, R. D. Boswell, Jr., Floyd Bowling, M. G. Boyce, C. L. Bradshaw, J. P. Brewster, G. M. Brown, J. W. Brown, W. Evelyn Brown, C. W. Bruce, B. F. Bryant, L. Virginia Carlton, H. P. Carter, R. S. Christian, D. H. Clanton, B. G. Clark, J. A. Cooley, C. L. Cope, R. J. Cormier, R. W. Cowan, H. J. Dark, J. M. Doran, L. A. Dye, E. D. Eaves, F. A. Ficken, M. K. Fort, Jr., Tomlinson Fort, Emma W. Garnett, R. L. Gay, I. C. Gentry, S. T. Gormsen, W. W. Graham, E. H. Hadlock, O. G. Harrold, Jr., T. W. Hildebrandt, A. T. Hind, Jr., J. H. Hoelzer, A. S. Householder, G. B. Huff, J. A. Hyden, John Jones, Jr., F. W. Kokomoor, Stephen Kulik, H. T. LaBorde, J. W. Lagrone, C. G. Latimer, H. L. Lee, D. B. Lowdenslager, G. H. Lundberg, Elna B. McBride, J. B. McGrory, S. W. McInnis, E. J. McShane, R. C. Meacham, G. W. Medlin, E. P. Miles, Jr., D. D. Miller, R. H. Moorman,

J. A. Nohel, J. D. Novak, C. R. Partington, G. L. Pate, P. B. Patterson, Lillian G. Perkins, C. G. Phipps, R. B. Plymale, Ellen F. Rasor, L. T. Ratner, H. A. Robinson, J. U. Russell, E. B. Shanks, D. C. Sheldon, A. R. Sloan, H. L. Slonecker, Jr., C. B. Smith, W. S. Snyder, D. E. South, F. W. Stallard, E. L. Stanley, L. W. Stark, W. G. Stokes, W. L. Strother, J. M. Thomas, C. W. Thomson, J. Clifton Thurman, A. W. Tucker, R. Z. Vause, Jr., Ruric E. Wheeler, M. C. Wicht, W. L. Williams, R. A. Willoughby, R. L. Wilson, F. J. Witt, M. D. Woodard, F. L. Wren, B. K. Youse.

The following officers were elected for the coming year: Chairman, Professor R. H. Moorman, Tennessee Polytechnic Institute; Vice-Chairman, Professor D. F. Barrow, University of Georgia; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *Three dimensional relaxation*, by Professor Ray Kinslow, Tennessee Polytechnic Institute, introduced by the Secretary.

The method of systematic relaxation of constraints has been found very useful in solving two-dimensional and axially symmetrical field problems. A technique is described by which the relaxation method may be applied to general three-dimensional fields. The relaxation net, which is cubical, is drawn oblique in such a manner that the computer need not visualize the three dimensions, but may perform the operations as if a hexagonal net had been employed in a plane-potential problem. A problem involving Laplace's equation is illustrated.

2. *Transposition theorems for linear equations and inequalities*, by Professor A. W. Tucker, Princeton University (By invitation).

In euclidean vector $n$-space the closed convex set $\{U \mid U = A_1x_1 + \cdots + A_px_p + B_1y_1 + \cdots + B_qy_q;\ x_i \geqq 0, \sum x_i = 1;\ y_j \geqq 0\}$ contains $U = 0$ *or* a shortest vector $\neq 0$ that makes an acute angle with each $A_i$ and a nonobtuse angle with each $B_j$. So, $AX + BY = 0$ has a solution $X \geqq 0, \sum x_i = 1$, $Y \geqq 0$ *or* $U'A > 0$, $U'B \geqq 0$ has a solution $U$ [Motzkin, Thesis, 1936]. Cases: (1) $B$ vacuous [Gordan, *Math. Ann.*, vol. 6, 1873, Stiemke, *Math. Ann.*, vol. 76, 1915; Dines, *Bull. A.M.S.* vol. 42, 1936]; (2) $A = -B_0$ (a vector) [Farkas, *Crelle*, vol. 124, 1902; Weyle, *Comm. Math. Helv.*, vol. 7, 1935]; (3) $B = I$ (identity matrix) [von Neumann-Morgenstern, *Theory of Games*, 1944]; (4) $A = S + I$, $B = (S, I)$, $S$ a skew-symmetric matrix [new sharpened existence theorem for games and linear programming].

3. *Mathematics for computers*, by Dr. A. S. Householder, Oak Ridge National Laboratory.

Suggestions are made for modifying the content of conventional courses in mathematics which, without detriment to the mathematical quality of the courses, would provide better preparation for students in the sciences and for those who will find employment as mathematicians in industrial and governmental laboratories. These suggestions include the development of analytical geometry in terms of matrices, the parallel development in calculus of differentials and of finite differences, of analytical and of numerical quadrature, of differential and of difference equations, and the introduction of the qualitative theory of equations with de-emphasis of closed solutions.

4. *On the equivalence of the usual definitions of conic sections*, by Professor G. B. Huff, University of Georgia.

In the standard textbooks on analytic geometry it is customary to begin by defining a conic section as a section of a right circular cone by a plane. The parabola, ellipse, and hyperbola are then defined again and studied as plane loci. Professor Huff pointed out the existence of simple figures which permit heuristic proofs of the equivalence of the two kinds of definitions.

5. *Mathematical education behind the iron curtain*, by Professor Stephen Kulik, University of South Carolina.

The paper presents the system of elementary and university education in the U.S.S.R. and in the Ukraine in particular. Mathematical programs and methods of teaching and organization of research work in mathematics are discussed.

6. *A new course in fundamental concepts of mathematics*, by Professor R. H. Moorman, Tennessee Polytechnic Institute.

Tennessee Board of Education requires that all who are certified to teach in Tennessee shall have had three quarter hours in *Fundamental Concepts of Mathematics*. Professor Moorman has developed such a course around the meaning and history of *number, measurement, the function concept*, and *the nature of proof*. An attempt is made to explain mathematics to students majoring in elementary education, physical education, music, *etc.*, who have previously taken no mathematics in college and almost none in high school. The course is proving to be a good orientation into mathematics for general education students.

7. *Stability of the motion of an infinitesimal body*, by Professor W. L. Williams, University of South Carolina.

In this paper, the motion of an infinitesimal body, subject to the attraction of three bodies with finite masses, is studied with particular emphasis on the areas where the motion is stable.

8. *An inductive approach to the real number system*, by Professor C. G. Phipps, University of Florida.

It is fashionable in modern mathematics to emphasize the axiomatic method. Consequently authors seldom show the inductive approach to the axioms used. This report outlines such an approach to the axioms of the real number system as an aid in a better understanding of mathematical processes.

9. *Trigonometry for calculus students*, by Mr. R. D. Boswell, Jr., University of Georgia.

The sine and cosine functions are defined by the usual power series expansion. Using elementary theorems of calculus, the basic properties of these functions are proved.

10. *Algebra courses for juniors and seniors*, by Professor R. W. Ball, Alabama Polytechnic Institute.

In the curriculum for junior and senior mathematics majors, it is suggested that the specialized courses in the theory of equations, matrices and determinants, *etc.* be replaced by a reasonably abstract course in modern algebra. Among the advantages in this replacement are economy of course offering for schools with a limited graduate program, a workable description of the nature of numbers for those who plan to teach arithmetic and beginning algebra, an early introduction to the notion of a mathematical proof, and a course which is often pleasantly different from the previous courses taken by the student.

11. *Excruciating problems*, by Professor R. A. Willoughby, Georgia Institute of Technology.

A number of examples of freshman and sophomore problems are presented. These problems tend to trap students who are sloppy in their analysis of problems and who depend blindly on memorized procedures.

12. *Invariants of certain linear differential equations*, by Professor R. W. Cowan, University of Florida.

Coefficients of the normal form of several linear differential equations are shown to be invariant under a linear transformation of the dependent variable. A proof by mathematical induction is supplied to enable one to determine simply the adjoint equation. Relations are established involving invariants and the coefficients of the adjoint equation.

13. *Note on Liapounov's second method*, by Professor J. A. Nohel, Georgia Institute of Technology.

Consider the system of nonlinear differential-functional equations (1) $x' = f(t, X(t))$, $' = d/dt$, where $x$ and $f$ are real vectors with $n$ components, $t$ is a real scalar, and $X(t)$ denotes the set of functions $\{x_k[u_k(t)]\}$, $(k = 1, \cdots, n,\ k = 1, \cdots, m)$. The functions $f_j$, $j = 1, \cdots, n$, and $u_k$, $k = 1, \cdots, m$, are given. In addition to assumptions which guarantee existence and uniqueness of solutions (see J. Franklin, "*On the existence of solutions of systems of functional differential equations*," Proc. of the A.M.S., vol. 5, 1954) suppose that $f(t, 0) = 0$; then $x = 0$ is a solution of (1). Sufficient conditions for the stability and asymptotic stability of the zero solution are given using a slight generalization of Liapounov's "Second Method" for ordinary differential equations.

14. *Differential equations with interfaces between regions*, by Mr. C. L. Bradshaw, Oak Ridge National Laboratory.

Reactor calculations in two or three dimensions give rise to the mathematical problem of determining the fundamental eigenvalue of a system of equations. A simple problem in two dimensions and two regions, where complete analytic solutions are possible, has been solved. Iterative methods for solving this problem have been evaluated as to rapidity of convergence and ease of preparation for a high speed computer.

15. *What is a solution of a differential equation?*, by Professor Tomlinson Fort, University of South Carolina.

In this paper the author considers the "solutions" usually given in text-books. He shows that not only are definitions frequently carelessly worded but that many functions not mentioned in the more commonly used text-books must be considered solutions under the definitions that the authors do give.

16. *A note on uniform spaces, by* Mr. B. F. Bryant, Vanderbilt University.

Let $X$ be compact uniform space which is not metrizable, and let the collection $\{V_\alpha\}$ be the subsets of $X^2$ which determine the topology of $X$. It was shown that if $f$ is a self-homeomorphism of $X$, then for each $V_\alpha$ there exists $(x_\alpha, y_\alpha)$ such that the orbit of $(x_\alpha, y_\alpha)$ under the homeomorphism $g(x, y) = (f(x), f(y))$ is contained in $V_\alpha$. This implies that if $X$ is a compact uniform space on which it is possible to define an unstable self-homeomorphism, then $X$ is metrizable.

17. *A note on vector spaces*, by Mr. R. H. Ackerson, Alabama Polytechnic Institute.

Let $V$ be a vector space of dimension $n$ over a field $F$, and $A$ a linear transformation of $V$ into itself. Let $R(A)$ denote the range of $A$, with dimension $r(A)$; and $N(A)$ denote the null space of $A$. The following theorem is proved: Let $I(A) = R(A) \cap N(A)$. The dimension of $I(A) = r(A) - r(A^2)$. The result is obtained without use of canonical forms of matrices. Some corollaries are given, including the equivalence of the three conditions $V = R(A) \oplus N(A)$, $I(A) = 0$, and $r(A) = r(A^2)$.

18. *Essential mappings*, by Professor M. K. Fort, Jr., University of Georgia.

Let $A$ be a finite subset of the plane $P$. Then there exists a continuous mapping $f$ of a circle

$S$ into $P-A$ such that $f$ is not homotopic to a constant in $P-A$, but such that for every proper subset $B$ of $A$, $f$ is homotopic to a constant in $P-B$.

19. *Mappings with property "T,"* by Professor E. B. Shanks, Vanderbilt University.

A triple of real numbers has property "T" provided each one of the triple is the number of units in the length of one side of some fixed triangle (such a triple will be called a triangle of real numbers). A mapping of the set of positive real numbers into itself has property "T" provided each triangle of real numbers is mapped into a triangle of real numbers. Results were presented relative to the class of mappings so defined.

20. *What is linear programming?*, by Professor F. A. Ficken, University of Tennessee.

A "transportation problem" gives warehouses $\alpha(\alpha=1, \cdots, m)$ with capacities $w_\alpha$, factories $j(j=1, \cdots, n)$ with products $p_j$, and unit shipment costs $c_{\alpha j}$ to $\alpha$ from $j$, and requires shipments $s_{\alpha j} \geqq 0$ such that $\sum_j s_{\alpha j}=w_\alpha$, $\sum_\alpha s_{\alpha j}=p_j$, and the total shipping cost $\sum_{\alpha,j} c_{\alpha j}s_{\alpha j}$ shall be a minimum. The general problem of linear programming requires an extreme value of a linear functional subject to a given system of linear constraints (equalities or inequalities). The object here is to illustrate the problem and in extremely simple terms to sketch and illustrate the "simplex method" of arranging calculations leading to a solution.

21. *Committee on the undergraduate mathematics program*, by Professor E. J. McShane, The University of Virginia. (By invitation.)

Professor McShane reported on the earlier meetings of the Committee on the Undergraduate Mathematical Program and on the material prepared by the group working last summer in Lawrence, Kansas. He also described the nature of Part II of the experimental text materials, now in preparation at Tulane University, and gave some examples of the type of applications to genetics, political science, *etc.* which it will contain.

22. *An orthotropic plate with a parabolic notch*, by Professor C. B. Smith, University of Florida.

A large orthotropic plate contains a notch in the form of a parabola with its principal axis parallel to an axis of elastic symmetry of the plate. The plate is taken to be in a state of plane stress under the action of forces applied along the boundary of the notch. The resulting stress distribution is discussed theoretically by using functions involving two complex variables.

23. *A note on functional completeness II*, by Professor R. A. Willoughby, Georgia Institute of Technology.

Given the set $S=\{0, 1, 2, 3\}$ and the operation $\times$ such that $a\times 0=0\times a=0$, $a\times 1=1\times a=a$, $a\times a=a$, $2\times 3=3\times 2=0$, there does not exist a $0\to 1$ cyclic permutation $\frown$ on $S$ for which $(S, \times, \frown)$ is functionally complete.

24. *A basic set of polynomial solutions for the Euler Poisson Darboux equation*, by Professors E. P. Miles, Jr. and Ernest Williams, Alabama Polytechnic Institute, presented by Professor Miles.

The authors consider the Cauchy Problem for the E.P.D. equation: $u(x_1, x_2, \cdots, x_m, 0)=f(x_1, x_2, \cdots, x_m)$, $u_t(x_1, x_2, \cdots, x_m, 0)=0$, $L_k[u]=\sum_{i=1}^m u_{x_ix_i}-u_{tt}+kt^{-1}u_t=0$. If $k>0$, $f$ is analytic, and a unique analytic solution exists, the homogeneous polynomials of degree $n$ in the expansion of $u(x_1, x_2, \cdots, x_m, t)$ are linear combinations of the following set of solutions for $L_k(u)=0$. For any set of non negative integers $b_j$ such that $b_0=0$, $\sum_{j=0}^m b_j=n$ let

$$H^{k,n}(k, t, x_1, \cdots, x_m) = \sum \left[ n!(a_0/2)! \Big/ \prod_{j=0}^{m} a_j![(b_j - a_j)/2]! t^{[a_0]} \prod_{j=1}^{m} x_j^{a_j} \right]$$

where $t^{[a_0]} = [1 \cdot 3 \cdot 5 \cdots (a-1)/(1+k)(3+k) \cdots (a-1+k)]t^{a_0}$ and the summation extends over all $a_j$ such that (1) $a_j \equiv b_j$, mod 2, $j=0, 1, 2, \cdots m$, (2) $\sum_{j=0}^{m} a_j = n$, and (3) $a_j \leqq b_j$, $j=1, 2, \cdots m$. For $k=0$ these polynomials form a portion of the authors' basic sets for the wave equation.

25. *A theorem on disks*, by Professor O. G. Harrold, Jr., University of Tennessee.

Let $J$ be a topological line (unbounded in both directions) in three-space. Does there exist a disk (closed, topological 2-cell) meeting $J$ in a single point whose boundary links $J$? It is proved that if there is some homeomorphism $h$ of space on itself such that the image of some pair of points on $J$ defines a rectifiable sub-arc of $h(J)$, such a disk exists, even a tame one.

26. *A sound mathematics curriculum for a well-staffed four year liberal arts college*, by Professor S. T. Gormsen, Rollins College.

Professor Gormsen led a general discussion relative to courses which all mathematics and science majors should take and the mathematics courses recommended for majors in other fields. It was agreed that more mathematics on all levels, and not less, should be taught. We should give more 20th and less 17th century mathematics. Linear algebra is as important perhaps as calculus.

27. *Generalized limit procedures*, by Professor E. J. McShane, The University of Virginia. (By invitation.)

This talk was devoted to the presentation of a theory of limits which is of considerable generality and at the same time is intended to be easily understood by beginners. A "system of stages" is a collection of one or more sets, each having one or more points in it, such that if $S_1$ and $S_2$ are members of the system, some set $S_3$ of the system is contained in both $S_1$ and $S_2$. A "limit process" consists of a function $f$ and a system $\mathfrak{S}$ of stages in the domain $D$ on which $f$ is defined; if $f$ is real-valued, it has $L$ as limit if for every $\epsilon > 0$, there is a stage $S$ in the system $\mathfrak{S}$ such that $|f(x) - L| < \epsilon$ for every $x$ in $S$. The usual theorems on limits of sums, *etc.*, can be proved easily. The theory covers the applications to sequences, series, integrals, *etc.*, without mental contortions.

28. *A simple proof of the Euler transformation*, by Professor Tomlinson Fort, University of South Carolina.

The Euler transformation of a convergent infinite series is formally established by means of the summation by parts formula. The remainder after $n$ terms appears in a form which can be shown to approach zero in two or three lines.

29. *On a series for* $(1+k)^k$, by Professor D. E. South, University of Florida.

Using the moment generating function, one can show that the $k$th moment about the origin of a Bernoulli distribution has the form $\sum_{i=1}^{k} A_{ki} r^{(i)} p^i$. In order to determine the coefficients $A_{ki}$, the fundamental relation between the operators $\Delta$ and $E$ of the calculus on finite differences and the theorem on the $n$th difference of a polynomial were used to show that $(1+k)^k$ can be expressed as

$$\sum_{i=0}^{k} (-1)^{i+2} \binom{k+1}{i+1} (k-i)^k, \quad k \text{ being a positive integer.}$$

30. *Machine computation of definite integrals*, by Mr. C. L. Gerberich, Oak Ridge National Laboratory, introduced by the Secretary.

Very often in the field of scientific computations there will arise definite integrals that will not

lend themselves to an analytic treatment. It is therefore necessary to pick from among the varied numerical methods those which will best fit the needs of a high speed digital computer. Of the methods studied it was found that Gauss's quadrature formula is by far the superior as a general purpose method. The only type problems that show the Gauss method at a disadvantage are those with integrands given only at equally spaced points. Such problems may arise when the integrand is given by experimental data.

31. *Computing higher mathematical functions*, by Mr. S. G. Campbell, Oak Ridge National Laboratory, introduced by Dr. A. S. Householder.

The most generally useful analytic methods for calculating higher mathematical functions are series methods (power series, asymptotic series, hypergeometric series, *etc.*), approximating polynomials (particularly Legendre and Tchebycheff polynomials), and continued fractions. Numerical methods include interpolation, successive approximation, numerical quadrature, and numerical solution of differential equations. None of these methods possesses over-all superiority. The use in modern science of increasingly complicated higher mathematical functions, often with complex argument, and of high speed digital computers to evaluate such functions, indicates that an increasing percentage of students who become professional mathematicians will need to know and be able to evaluate these computational techniques.

32. *An involutorial transformation of order 5 associated with a pencil of twisted cubics*, by Professor L. A. Dye, The Citadel.

A 1-1 correspondence between a pencil of twisted cubics and a pencil of quadric surfaces is used to determine an involutorial cremona transformation of order 5. The base of the web of homoloids consists of two twisted cubics, one of which is double, and 5 simple lines, three of which are parasitic.

33. *A note on starlike functions*, by Professor M. C. Wicht, North Georgia College.

The author applies Royster's condition for starlikeness (*Duke Journal*, September, 1952) of mappings with respect to several families of directed curves including the solutions of $\dot{z}-Pz=Q$, where $\dot{z}=dz/dt$, $P=P(t)$, $Q=Q(t)$ and $z=x(t)+iy(t)$ with $x(t)$ and $y(t)$ real. The condition reduces to

$$(Px+Q)S_h+yS_v=0$$

where $S_h=$Real Part $[-iF'(z)/F(z)]$ and $S_v=$Real Part $[F'(z)/F(z)]$. The relations between this condition for starlikeness on these families and those for starlikeness on families of vertical and horizontal lines was noted.

34. *On matrices all of whose characteristic roots lie within the unit circle*, by Professor H. T. LaBorde, The University of the South.

A problem arising in economics is one of determining sufficient criteria that a given matrix have all of its characteristic roots in or on the unit circle. If $A=(a_{ij})$ is a real square matrix of order $n$, let $R_i=\sum_{j=1}^{n}|a_{ij}|$, $S_i=\sum_{j=1}^{n}|a_{ji}|$ $(i=1, 2, \cdots, n)$. It is proved here that a matrix can have as many as one-half the $R_i$ and one-half the $S_i$ greater than unity and still have all of its characteristic roots lie within the unit circle.

35. *On the construction of a ternary quadratic form*, by Professor E. H. Hadlock, University of Florida.

Given the invariants $\Omega$ and $\Delta$ associated with a ternary quadratic form, let certain restrictions be imposed on the first coefficient of the form and the third coefficient of its reciprocal form. Then

formulas are obtained for each of the five remaining coefficients of the form and also for each of the five remaining coefficients of the reciprocal form.

36. *Lipschitz conditions for a piece-wise smooth function*, by Professor W. S. Snyder, University of Tennessee.

Professor Snyder demonstrated the theorem: Let $R$ be a convex body in euclidean $N$ space, $E^N$, and $f$ a real-valued continuous function defined on $R$. Let $S$ be a Lipschitzian surface in $E^N$, *i.e.*, $S=G(Q)$ where $G$ is a Lipschitzian vector function defined from the unit cube $Q$ of $E^{N-1}$ to $E^N$. Assume there is a constant $M$ such that at each $x\in R-S$ $f$ has first partial derivatives whose absolute values do not exceed $M$. Then $f$ satisfies a Lipschitz condition in $R$ in the sense that if $L$ is a segment and $L\subset R$ then on $L$, $|\Delta f| \leqq M|L|\sqrt{N}$.

37. *A geometric approach to the derivation of formulae for derivatives of an analytic function using polar coordinates*, by Professor S. T. Gormsen, Rollins College.

This paper illustrates a rigorous geometric approach to the derivation of the one-directional formulae for the derivatives of analytic functions using polar coordinates. As a biproduct a direct derivation of the necessary Cauchy-Riemann conditions are established.

38. *On some questions related to convexity*, by Professor L. T. Ratner, Vanderbilt University.

Several theorems pertaining to convexity of functions and of sets are generalized and extended to include approximate convexity. Sets which are approximately convex according to either of two definitions are characterized in new terms, and the exact relationship between the two definitions of approximate convexity is determined. Finally, a study is made of local approximate convexity.

39. *Subsets of Peano spaces*, by Professor W. L. Strother, University of Miami.

By $S^1(X)$ Professor Strother denotes the space of non-null closed subsets of the space $X$ and for $n>1$ he defines $S^n(X)$ to be the space of non-null closed subsets of $S^{n-1}(X)$. He establishes the theorem: If $X$ is a Peano space, then $S^n(X)$ is a Peano space.

H. A. ROBINSON, *Secretary*

## THE APRIL MEETING OF THE IOWA SECTION

The forty-second annual meeting of the Iowa Section was held jointly with the sixty-seventh annual meeting of the Iowa Academy of Science and the twenty-second convention of the Junior Academy of Science at St. Ambrose College, Davenport, Iowa, on April 15 and 16, 1955. The Chairman, Professor H. T. Muhly, the Vice-Chairman, Professor F. A. Brandner, and Reverend T. J. Taylor presided in turn. The officers elected are: Chairman, Professor F. A. Brandner, Iowa State College; Vice-Chairman, Professor R. S. Jacobsen, Luther College; and Secretary, Professor Fred Robertson, Iowa State College.

The registered attendance was 39 including the following 28 members:

H. G. Apostle, Barbara J. Beechler, R. H. Bing, F. A. Brandner, E. L. Canfield, J. O. Chellevold, E. W. Chittenden, Byron Cosby, Jr., A. T. Craig, W. M. Davis, Kathryn P. Ellis, J. J. L. Hinrichsen, R. V. Hogg, Rev. J. A. Hratz, L. A. Knowler, O. C. Kreider, H. T. Muhly, E. N. Oberg, Fred Robertson, W. C. Ross, Jr., Hazel M. Rothlisberger, D. E. Sanderson, J. A. Schumaker, R. C. Seber, F. M. Stein, H. P. Thielman, H. J. Weiss, Roscoe Woods.

The following papers were presented:

1. *The approximate solution of integro-differential equations*, by Mr. F. M. Stein, Iowa Wesleyan College.

It is known that, under certain conditions, the integro-differential equation

$$U(u) \equiv L(u) - \int_a^b h(x, t)u(t)dt = f(x)$$

with boundary conditions $U_i(u)=0$, $(i=1, 2, \cdots, m)$ has a unique solution of the form $u(x) = \int_a^b H(x, t)f(t)dt$, where $H(x, t)$ is the Green's function of the problem. This paper examines the existence, uniqueness and conditions for convergence of a polynomial which satisfies the boundary conditions and the criterion that

$$\int_a^b |f(x) - U[P_n(x)]|^r dx$$

shall be the least for $r>0$. An analogous problem is also examined when the approximating function is a trigonometric sum. When $r=2$ the coefficients are actually determined.

2. *Bending of a rectangular plate with even and odd order boundary conditions*, by Mr. J. P. Li, Iowa State College, introduced by the Secretary.

Bending of a rectangular plate with two adjacent edges clamped can be solved systematically by Fourier method. The deflection of the plate is taken in the form $w=w_1+w_2$ where $w_1$ is the deflection of a simply supported plate and $w_2$ represents a correction state which is chosen as two infinite series of particular solutions of the biharmonic equation. By adjusting the values of coefficients of $w_2$ by ordinary Fourier analysis, all of the specified boundary conditions can be completely satisfied. Examples are worked out for a rectangular plate subjected to uniformly distributed load with two adjacent edges clamped and the others either simply supported or free.

3. *Beams of uniform strength subjected to uniformly distributed loads*, by Mr. J. P. Li and Mr. W. A. Gross, Iowa State College, presented by Mr. Gross, who was introduced by the Secretary.

Beams of uniform strength in bending have the same maximum flexural stress at any cross section. Shapes, deflections, and weights for cantilever, simple, and fixed beams of uniform strength are derived taking into account the weights of the beams. Explicit solutions are given for certain beams of rectangular cross sections and constant height or constant width.

4. *A network for representing elastic bodies in spherical coordinates*, by Mr. W. A. Gross, Iowa State College.

A network representing elastic bodies having linear stress relations and described in spherical coordinates is developed. The case of axial symmetry is treated. Equilibrium is satisfied by a finite difference grid which assures compatibility of strain. Usual boundary conditions may be fitted.

5. *A remark on integrally closed local domains*, by Professor H. T. Muhly, State University of Iowa.

Let $L$ be a complete two dimensional local domain which is integrally closed in its quotient field, and let $m$ be the ideal of non-units in $L$. Assume that $L$ contains a field $k$ over which the residue field $L/M$ is finite and algebraic. Under these assumptions it follows that if $x$, $y$ is any system of parameters for $L$, then $L$ is a regular module over the power series ring $k\{x, y\}$ in the sense that $L$ admits an independent basis over $k\{x, y\}$.

6. *A note on function spaces*, by Professor H. A. Dye, State University of Iowa, introduced by the Secretary.

This is to call attention to a theorem, characterizing compact spaces in terms of allied structures, which does not seem to be stated in the extensive literature on this subject. For $X$ a compact Hausdorff space, denote by $P(X)$ the (multiplicative) group of all positive continuous functions on $X$. Call a group isomorphism $\phi$ of $P(X)$ on $P(Y)$ bounded if, under both $\phi$ and its inverse, the unit sphere is transformed into a bounded set. The theorem asserts that each bounded isomorphism between $P(X)$ and $P(Y)$ is implemented (in the natural sense) by a homeomorphism of $X$ on $Y$.

7. *On testing hypotheses about a certain type of truncated distribution*, by Professor R. V. Hogg, State University of Iowa.

Let the results of an experiment be described by the known positive density $g(x)$. Due to some limitations, possibly physical, we are able only to observe results of the experiment between two fixed, but unknown, values, say $c$ and $d$. We consider two hypotheses: 1) that $m_0$ is the median of this truncated distribution; and 2) that $c=c_0$, $d=d_0$. We test these hypotheses against all possible alternatives by use of the likelihood ratio $\lambda$. If $n$ is our sample size, we find that for the first hypothesis the principal $n$th root of $\lambda$, say $t$, has the density $(n-1)t^{n-2}$, $0<t<1$, if the hypothesis is true. In the second case, $t$ has the density $n(n-1)t^{n-2}(1-t)$, $0<t<1$. In both instances, the interesting fact to observe is that while the computation of $t$ depends upon the known density $g(x)$, the distribution of $t$ is free of this density.

8. *A property of the median*, by Professor A. T. Craig, State University of Iowa.

In a regular case of estimation, let a random variable have non-degenerate probability density which depends on a parameter $\theta$ for which there exists a sufficient statistic. It is proved that the probability density of the median of a random sample cannot be free of $\theta$. Thus the median and the sufficient statistic are always stochastically dependent.

9. *Some remarks on the December Meeting of the Board of Governors of the Mathematical Association of America*, by Professor E. N. Oberg, State University of Iowa (Section Governor).

10. *Designing a mathematics curriculum for students of the non-physical sciences*, by Mr. Robert Seber, State University of Iowa. (By invitation.)

We consider a mathematics curriculum with respect to (I) topical content, (II) style of presentation, and (III) illustrative material. Attention is called to two types of course sequences built about a fixed core of topics and denoted as "traditional" and "unified." Dissatisfaction with current curricula is centered upon (II) and upon the non-inclusion and non-emphasis of topics not represented in the core.

We view a curriculum as a collection of statements presented verbally and symbolically. Certain terms of mathematics are the referents of terms used in the non-physical sciences. It is in this sense that mathematics is applied in these sciences. This application is considered in two categories: (A) quantitative description of phenomena of non-physical science and (B) mathematical models to schematize non-physical science phenomena.

A course sequence with a prerequisite of college algebra and trigonometry is proposed and considered relative to (I), (II), and (III). This sequence includes topics of algebra, geometry, analysis, and statistics. The course structure is classified as "unified." The standard core of topics is retained but is not treated as extensively. Emphasis is placed on the style of presentation which is for each new topic on "explication" in the sense of Carnap. Topics of modern algebra, statistics and probability are an integral part of the course. Each topic is considered with reference to its descriptive

background, formalization as a mathematical system, and application in describing phenomena of non-physical science. Types of illustrative material relative to applications are presented in detail for some topics.

11. *Developing new mathematics*, by Professor R. H. Bing, University of Wisconsin. (By invitation.)

This discussion emphasizes the fact that mathematics is an alive and growing subject. The following things are considered: examples of new mathematics recently discovered, unsolved problems that are being attacked, the amount of new mathematics being developed, experiences of researchers in developing new mathematics, the thrill that students experience in inventing mathematics, the role of teachers in leading students to see the beauty of mathematics.

FRED ROBERTSON, *Secretary*

## THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Georgetown College, Georgetown, Kentucky, on April 30, 1955. Professor Charles Hatfield, Chairman of the Section, presided at both the morning and afternoon session.

There were 49 persons in attendance, including the following 32 members of the Association:

R. C. Brown, Jr., H. W. Burnette, Esther A. Compton, J. B. Cornelison, V. F. Cowling, H. H. Downing, J. C. Eaves, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Tadeusz Leser, E. J. Mickle, W. L. Moore, R. S. Park, Sallie E. Pence, Mary Pettus, J. D. Riley, G. G. Roberts, W. J. Robinson, D. C. Rose, M. I. Rose, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, K. E. Stoll, Louise C. Stolle, W. C. Swift, J. T. Vallandingham, J. A. Ward, T. M. Wright, W. M. Zaring.

At the business meeting the following officers were elected for the coming year: Chairman, Professor V. F. Cowling, University of Kentucky; Secretary-Treasurer, Professor A. W. Goodman, University of Kentucky; Traveling Lecturer, Professor J. C. Eaves, University of Kentucky.

The following papers were presented:

1. *Isometric mappings*, by Professor A. W. Goodman, University of Kentucky.

Definitions were given of metric space, Hilbert space, and isometric mapping, and an example was given of a space of four points which cannot be mapped isometrically on any sub-space of Hilbert space.

2. *Connected ideals in compact connected mobs*, by Professor W. M. Faucett, University of Kentucky, introduced by the Secretary.

A mob is a Hausdorff space that admits a continuous associative multiplication. It is known that if a mob $S$ is connected and possesses a minimal ideal $K$, then $K$ is connected. The purpose of this note is to state the result that if $S$ is compact, connected, and $S^2 = S$, then every maximal ideal of $S$ is connected. An example is given to show that with the same hypothesis $S$ may contain disconnected ideals.

3. *Heredity in mathematics*, by Professor R. W. Bagley, University of Kentucky.

Some elementary notions concerning hereditarily separable spaces are discussed.

4. *On the product of linear forms*, by Mr. W. M. Zaring, University of Kentucky.

Let $a_{ij}$ and $b_i$, $i, j = 1, \cdots, n$ be real numbers. It has been conjectured that

$$\left| \prod_{i=1}^{n} (L_i(X) - b_i) \right| \leq \frac{|D|}{2^n}$$

has integral solutions $X_i$, $i = 1, \cdots, n$, where $D = |a_{ij}| \neq 0$ and $L_i(X) = \sum_{j=1}^{n} a_{ij}X_j$. Minkowski first proved the result for $n = 2$. The author presents R. Bellman's proof (*A note on the product of linear forms*, this MONTHLY, vol. 51, 1944, p. 161) that the conjecture is true if the $a_{ij}$'s form an orthogonal transformation.

5. *The zeros of polynomials*, by Professor V. F. Cowling, University of Kentucky.

In this paper a discussion is given of the various types of problems concerned with the distribution of the zeros of polynomials. Particular emphasis is placed on those results in which limitations on the distribution of the zeros are obtained respectively in terms of the moduli and arguments of the coefficients.

6. *The Gaussian integral factors of* $10i$, by Mr. H. W. Burnette, University of Kentucky.

The speaker discusses, briefly, the integral factors of composite rational integers, such as 24, pointing out which factors are non-trivial. Then he defines the necessary quantities to show how Gaussian integers of the form $a+bi$, where $a = 0$, in particular, $10i$, may be factored into 17 essentially different integral factorizations.

7. *The Silverman-Toeplitz theorem*, by Mr. W. C. Swift, University of Kentucky.

The Silverman-Toeplitz theorem is a well known result. Except for one complication, the proof is straightforward. Aided by an example, the method of resolving this difficulty is suggested. An application illustrates the theorem.

8. *Modern teaching of survey courses*, by Professor Arno Jaeger, University of Cincinnati, introduced by Professor G. M. Merriam, University of Cincinnati.

This is a report on an elementary survey course for students who need mathematics as a college requirement only and not as a prerequisite for other subjects. The instructor felt very strongly that an introduction to modern mathematics is more suitable to demonstrate the nature of mathematics than a further elaboration of the naturally limited concepts of high-school mathematics. Even the freshmen understood the modern ideas and conceptions presented rigidly and often in Bourbaki's language, but with examples from ordinary life if possible. The students enjoyed making contributions by composing lecture notes, giving oral reports, or constructing visual aids.

9. *On Kasner's circle*, by Mr. M. I. Rose, University of Kentucky.

Let $w = u+iv$ be a polygenic function of the complex variable $z = x+iy$. Let the complex number $z+\Delta z$ approach $z$ along a line of slope $m$. Then Kasner showed that the points $\alpha(m)+i\beta(m)$

describe a circle in the derivative plane. In this paper we study an analogous problem for the case in which $w$ is a function of a hypercomplex variable $z=1x+iy$, having a multiplication table

| | 1 | $i$ |
|---|---|---|
| 1 | 1 | $i$ |
| $i$ | $i$ | $a+bi$ |

and find that the set of points $\alpha(m)+i\beta(m)$ describe a conic section determined by $a$ and $b$.

10. *A summer school program for in-service teachers*, by Professor J. C. Eaves, University of Kentucky.

In most cases in-service teachers must obtain their mathematics training in summer classes. Some of the courses available to them are courses that they have already had; other courses are strictly at the research level and the participant finds that he has insufficient background to profit from them. This is an attempt to combine topics from all of the various courses, selecting from both classical mathematics and modern mathematics. This will give the teacher studying in the summer a broader coverage of interesting material and will give him early in his training some idea of the modern concept of mathematics.

11. *Bounded variation and absolute continuity*, by Professor E. J. Mickle, Ohio State University. (By invitation.)

The definitions of bounded variation and absolute continuity for a real-valued function $y=f(x)$ were given. The generalization of these concepts by Tonelli to a function of two variables $z=z(x, y)$ were discussed. Examples were given to illustrate the difficulties encountered in generalizing these concepts to plane mappings. The results of Radó and Cesari in giving a complete generalization of bounded variation and absolute continuity were stated.

A. W. GOODMAN, *Secretary*

## THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Morgan State College, Baltimore, Maryland, on April 16, 1955. The Section joined with the American Chemical Society (Maryland Section), National Institute of Science, Beta Kappa Chi Scientific Society, Maryland Conference of Biology Teachers, and the American Association of Physics Teachers, in the Dedication Program for Calloway Hall and the new Science Quadrangle at Morgan State College. Professor C. H. Frick, Chairman of the Section, presided at the morning session of the Section.

There were 48 persons in attendance, including the following 40 members of the Association:

R. P. Bailey, T. J. Benac, R. O. Blummer, Jr., B. H. Buikstra, L. G. Campbell, P. L. Chessin, P. J. Federico, J. F. Foley, Gloria C. Ford, C. H. Frick, Michael Goldberg, R. A. Good, J. R.

Gorman, J. R. Hammond, Louise S. Hunter, C. F. Koehler, W. W. Leutert, R. M. Mason, Carol V. McCamman, L. I. Mishoe, T. W. Moore, G. A. Paxson, J. W. Popow, W. W. Proctor, R. S. Raven, R. W. Rector, Sister Rita, H. R. Smith, W. S. Soar, C. F. Stephens, M. F. Stilwell, W. J. Strange, E. G. Swafford, J. A. Tierney, M. M. Torrey, Marcelle M. Walker, P. M. Whitman, Beryl W. Williams, R. H. Wilson, Jr., A. W. Yonda.

The following officers were elected to serve for a period of one year: Chairman, Professor F. E. Johnston, George Washington University; Vice-Chairmen, Professor Ella C. Marth, Wilson Teachers College and Mr. W. H. Norris, Jr., Maury High School, Norfolk, Virginia; Secretary, Professor R. P. Bailey, U. S. Naval Academy.

Miss Carol V. McCamman reported that 81 high schools participated in the Second Annual High School Contest given April 6, 1955. Attention was called to the excellent results obtained by Mr. R. O. Blummer and Mr. Paul Chessin in the campaign to interest industrial organizations in the contest, which now boasts an extensive list of prizes. The Section approved an amendment to the By-Laws to include the Sectional Governor among the members of the Executive Committee.

The following papers were presented:

1. *Digitalization of war games*, by Dr. W. W. Leutert, Chief, Computing Laboratory, Aberdeen Proving Ground.

An outline was presented of how a battle may be simulated on a high speed digital computer.

2. *Rotors, plane and fancy*, by Mr. Michael Goldberg, Department of the Navy. (By invitation.)

The history and development of rotors, which are generalizations of curves and surfaces of constant width, was discussed. Meissner derived a Fourier series expression describing the totality of rotors touching all sides of the regular plane polygons. Mr. Goldberg described his kinematic methods for obtaining these rotors and showed how they are used for obtaining rotors in spherical polygons also. Meissner showed that non-spherical rotors exist for the tetrahedron, the cube and the octahedron, but not the dodecahedron and the icosahedron. The works of Euler, Minkowski, Fujiwara, Blaschke, Meissner, J. W. Green and others on these and related problems were discussed. A large collection of plane, spherical and solid models was used to illustrate the lecture.

At luncheon, the Section and the other scientific societies participating heard Dr. F. G. Watson of Harvard University speak on *A Laboratory for Learning*. The joint afternoon session was addressed by Dr. G. P. Harnwell, President of the University of Pennsylvania, who spoke on *Implications of Current Developments in Physics*.

R. P. BAILEY, *Secretary*

*Correction*: In the report of the Fall meeting of this section, there is an error in abstract 1A, this MONTHLY, vol. 62, p. 211; in the first line the symbol $(m/k)$ should read $\binom{m}{k}$.

## THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The fourteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Queens College, Flushing, New York, on April 30, 1955. Professor A. B. Brown, Collegiate Vice-Chairman of the Section, presided at the morning session and Professor H. F. Fehr, Chairman of the Section, presided at the afternoon session.

There were 116 persons in attendance, including the following 92 members of the Association:

Joseph Andrushkiw, R. G. Archibald, W. W. Bessell, Jr., S. I. Birnbaum, D. H. Blanksteen, Samuel Borofsky, C. B. Boyer, A. B. Brown, Azelle Brown, J. H. Bushey, Jewell H. Bushey, A. J. Carlan, Audrey M. Carlan, G. B. Charlesworth, Charles Clos, P. J. Cocuzza, H. J. Cohen, T. F. Cope, Demetrios Counes, W. H. H. Cowles, D. R. Davis, I. A. Dodes, J. N. Eastham, J. E. Eaton, W. H. Fagerstrom, H. F. Fehr, J. M. Feld, William Forman, R. M. Foster, Brother Linus R. Foy, D. H. Frank, E. T. Frankel, Leona Freeman, M. P. Friedman, Bernard Greenspan, Harriet M. Griffin, Laura Guggenbuhl, C. M. Hebbert, M. J. Hellman, G. C. Helme, E. Marie Hove, Joseph Jablonower, Aida Kalish, O. J. Karst, L. S. Kennison, G. A. Keyes, H. S. Kieval, A. E. Kinney, Charles Koren, A. T. Kovitz, A. W. Landers, C. H. Lehmann, C. B. Maile, Jr., J. H. Manheim, May H. Maria, F. H. Miller, A. J. Mortola, P. M. Moskowitz, C. J. Oberist, Eugene Odin, L. F. Ollmann, C. F. Pinzka, Walter Prenowitz, James J. Quinn, H. W. Raudenbush, C. F. Rehberg, Susan L. Reid, Selby Robinson, N. J. Rose, M. F. Rosskopf, H. D. Ruderman, J. P. Russell, John Salerno, Charles Salkind, A. H. Sarno, Abraham Schwartz, Aaron Shapiro, E. I. Shapiro, James Singer, Sister M. Anita, Sister Maria Loyola, Morris Smith, Mildred M. Sullivan, R. L. Swain, F. B. Taylor, M. Virginia Terhune, L. F. Tolle, Annita Tuller, R. M. Warten, M. E. White, J. M. Wolfe, Leo Zippin.

The following officers were elected for the year 1955–56: Chairman, Professor A. B. Brown, Queens College; Collegiate Vice-Chairman, Professor Mina S. Rees, Hunter College; High School Vice-Chairman, Dr. Barnet Rich, Brooklyn Technical High School; Secretary, Dr. Azelle Brown, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting reports were given by the secretary, the treasurer, the Committee on Contests and Awards, and the Committee on Coordinating Mathematical Training.

The following report on the activities of the Committee on Contests and Awards was given by its chairman, Professor W. H. Fagerstrom.

There were 881 schools registered for the sixth annual mathematics contest, which is sponsored by the Metropolitan New York Section. Of this number 479 schools registered with the Metropolitan New York Section and 402 schools registered with the eight other state units that operate independently, using the same questions and rules of the contest as used by the Metropolitan New York Section. These registrations were distributed as follows: British Columbia—50 schools, Colorado—65 schools, Illinois—85 schools, Manitoba—5 schools, Oregon—61 schools, Upper New York—46 schools, Washington—60 schools, and Wyoming—30 schools.

The 881 schools were from nearly every state in the union, three provinces of Canada, Hawaii, and Scotland. There were over 23,000 students enrolled for

the contest. The winning school was not announced, since all of the scores were not known. Anyone desiring information about the contest should write to Professor W. H. Fagerstrom, City College, New York 31, N. Y.

Professor F. H. Miller reported the following recommendations for the Committee on Coordinating Mathematical Training: 1) three years of high school mathematics as a minimum for all college entrants; 2) separation of college-entrance and other students in the high schools; 3) greater stress on concepts and their application to computational procedures.

The group voted that the Committee should notify the Upper New York State Section that it approved such recommendations.

Dr. J. J. Theobald, President of Queens College, welcomed the people at the meeting, and then the following papers were presented:

1. *What the high school can do to recruit teachers of mathematics*, by Mr. L. W. Schlumpf, Andrew Jackson High School, St. Albans, introduced by the Secretary.

The shortage of qualified teachers of secondary mathematics poses an acute problem for all future training in this field. The problem promises to become even more serious. More efficient use of available personnel should be obtained by effecting different license requirements for teachers performing the two different basic jobs of high school mathematics: teaching those who continue their studies in mathematics and teaching those who are not academically minded. Every effort should be made to arouse the interest of able high school pupils in mathematics through segregation of them in honor classes, instruction of them by special methods, assignment to them of superior and enthusiastic teachers, and establishment of a comprehensive extra-curricular program in this field.

2. *Recruitment of teachers of mathematics*, by Mr. Joseph Jablonower, Board of Examiners, City of New York.

The shortage of teachers of mathematics and of students who are preparing to teach mathematics is already acute. With the increase in school population, and the depletion in the teaching ranks, we face a situation that calls for radical measures. Administration and boards of education have been yielding to the temptation of lowering requirements in professional preparation for teaching and reducing requirements as to knowledge of the subject.

Industries and educational foundations should be alerted to the needs of the situation and should get states and municipalities to recognize education as a major and indispensable industry. Industry and educational foundations need to work out wisely subventions that will assure trainees for the teaching of mathematics. Boards of education need to formulate salary schedules which will assure the competent teacher a scale of living respected by the community.

3. *What the colleges can do to recruit teachers of mathematics*, by Professor D. R. Davis, State Teachers College, Montclair, New Jersey.

Experience suggests the following measures to promote an effective program for the recruitment of teachers of mathematics: 1) provide a good mathematics curriculum, 2) recommend and treat teaching as a profession which requires a minimum of four years of college, 3) maintain a recruiting program among the high schools with the cooperation of local teachers and administrators, guidance personnel, college faculty representatives and college student representatives, 4) secure positive assistance from state and local organizations for scholarships, 5) picture carefully the advantages of the teaching profession, 6) give effective supervision to those doing practice teaching, 7) work for better salaries, buildings, equipment, and working conditions, 8) help direct

the efforts of all concerned toward elevating the teaching profession, that people may choose it because they feel it has dignity, purpose, meaning, and the respect and support of the public.

4. *Hilbert's fifth problem*, by Professor Leo Zippin, Queens College.

At the International Congress of Mathematicians, Paris 1900 (see *Göttinger Nachrichten* of that year) Hilbert proposed a number of distinct research programs, called problems. What we now know as the Fifth Problem (certain other questions having been forgotten) is the following: *is a locally-euclidean group necessarily isomorphic to a Lie group?* The last fifty years have witnessed the affirmative solution of this problem as part of the analysis of the structure of locally compact groups. The roll call of the names of those who made significant contribution to this larger problem is too long to be given here.

5. *How to do by arithmetic what you cannot do with the calculus*, by Professor M. G. Salvadori, School of Engineering, Columbia University, introduced by the Secretary.

The difficulties encountered in the analytical solution of problems of great practical importance leads the applied mathematician to the use of approximate solutions obtainable often by elementary arithmetical tools. The talk illustrated the application of numerical methods to a variety of linear problems.

E. MARIE HOVE, *Secretary*

## THE APRIL MEETING OF THE MISSOURI SECTION

The annual spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at the University of Kansas City, Kansas City, Missouri, on April 22, 1955. Professor Maria Castellani, Chairman of the Section, presided at the morning session, and Reverend W. C. Doyle, Rockhurst College, presided at the afternoon session.

There were 45 persons in attendance, including the following 30 members of the Association:

John J. Andrews, S. Louise Beasley, C. A. Bridger, Maria Castellani, John F. Daly, W. C. Doyle, D. H. Erkiletian, Jr., C. V. Fronabarger, J. D. Haggard, Nola L. A. Haynes, F. F. Helton, N. Q. Hubbard, G. H. Jamison, C. A. Johnson, L. O. Jones, P. S. Jones, C. E. Kelley, P. G. Kirmser, S. L. Levy, F. H. Lloyd, Marie A. Moore, O. J. Peterson, L. E. Pummill, Lois J. Roper, J. S. Rosen, Robert Schatten, W. R. Scott, R. G. Smith, W. A. Vezeau, Margaret F. Willerding.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Francis Regan, St. Louis University; Vice-Chairman, Professor H. D. Brunk, University of Missouri; Local Secretary-Treasurer, Professor Marie A. Moore, Harris Teachers College. Professor Margaret F. Willerding, Harris Teachers College, retained her position of Association Secretary for a fourth year.

The following program was presented at the morning session:

1. *The significance and derivation of the formula*, by Mr. N. Q. Hubbard, Lincoln High School, Kansas City.

The study of the formula covers an extensive field in the branches of mathematics. The re-

search on this problem extends from the dates of 1857 to 1955. The significance and derivation of the formula, as it appears in print, covers an abundance of research work on this topic. The validity has been shown by various citations in this study by noted scholars in the field of mathematics whose authenticity has met the criteria in educational research. This study shows very plainly the aids, new methods of procedures, principles and modern avenues of approach to algebraic solution in secondary mathematics. The significance and derivation of the formula stands out as one of the basic concepts of junior and senior high school mathematics of today.

2. *A differential equation applicable to population problems*, by Mr. C. A. Bridger, Bureau of Vital Statistics, Jefferson City.

The differential equation $dP = f(P)F(t)dt$, where $f$ and $F$ are polynomials, is assumed. Examples for simple cases are developed. The Pearl-Reed logistic occurs when $f$ is a quadratic and $F$ is a constant. On the basis of information from previous decades, estimates by counties for Missouri are made for 1960. In over half of the counties, the exponential form of the solution of the equation is used.

3. *Birth, death, and waiting in line*, by Dr. Ernest Koenigsberg, Midwest Research Institute, Kansas City, introduced by the Secretary.

The problems of waiting in lines or queues can be introduced in terms of birth and death processes. A queue system is described by three characteristics: 1) a birth process or input mechanism; 2) a queue discipline (*e.g.*, "first come, first served"); 3) a death process or exit mechanism.

Several birth and death processes are described and formulated, and the "exponential holding time" queue system is formulated in terms of the rate of change of the number of units at the service station (*i.e.*, those waiting in line and those being served). The "constant holding time" queue system is also formulated in terms of the number of ways that a given number of units can be at the service station. Various other cases are discussed: queuing with priorities, queuing with special service, random queue disciplines, and now Poisson birth processes.

4. *Reorientation in economic theory: linear and non-linear programming*, by Professor E. Altschul, University of Kansas City, introduced by the Secretary.

*Theory of Games* by von Neumann and Morgenstern has initiated a new trend in developing analytical tools in economic theory and applied economics. Human actions geared toward optimization of various goals represent a maximum problem entirely different from that of physics. In a society each participant has to maximize a function without having control over all variables, being forced to meet actions of opponents.

Von Neumann and Morgenstern proved that economic problems are not those resembling maximum problems in physics, but problems of maximizing under constraint conditions, leading to the analysis of relative maxima as expressed by linear inequalities. The traditional approach of calculus must be replaced by matrix analysis. The new minimax approach opens a wide avenue for economic generalizations and development of flexible tools in solving practical problems of optimization in managerial decision.

5. *Some sign and rank tests in statistics*, by Professor W. A. Vezeau, St. Louis University.

A short discussion of basic concepts in statistics was presented as background. Then a historical development of sign and rank tests was given. Reasons for the use of such tests by mathematics teachers were presented with particular stress upon the introduction of such tests as special topics or projects in certain classes of mathematics. A number of practical problems were discussed using some of the sign and rank tests.

6. *Rational function approximations for the exponential function*, by Mr. Y. L. Luke, Midwest Research Institute, Kansas City, introduced by the Secretary.

Employing Chebyshev polynomials, simple rational function approximations for the exponential function in the complex domain are obtained. The technique follows that of C. Lanczos and is known as the $\tau$-method for the solution of linear differential equations with rational coefficients. Some numerical examples are present. Results are useful for stability and response studies of time delay control systems.

7. *Infinite symmetric groups*, by Professor W. R. Scott, University of Kansas.

Let $X$ and $Y$ be infinite cardinal numbers. The infinite symmetric group $S(X, Y)$ is the group of $1-1$ transformations of a set of order $X$ onto itself in which fewer than $Y$ elements are moved. The alternating group $A(X)$ is the subgroup of finite even permutations in $S(X, Y)$. Any homomorphism of any $S(X, Y)$ or $A(X)$ on to an $S(X, Y)$ or $A(U)$ is an isomorphism and $X = U$, $Y = V$. Any factor group of $S(X, Y)$ contains a subgroup isomorphic to $S(X, Y)$ if $Y \neq \aleph_0$. If $2^Z < X$ ($Z$ may be finite) and $Y \neq \aleph_0$, then neither $S(X, Y)$ nor $A(X)$ contains a subgroup of index $\leq Z$.

8. *Phase plane solution of non-linear differential equations*, by Dr. S. L. Levy, Midwest Research Institute, Kansas City.

The ordinary electric bell's motion is analyzed in terms of an on-off servo-system. The analysis is completely graphic and is, furthermore, fully idealized so that it does not detract from the physical principles involved and the use of the phase-plane method. The response of a more realistic system is indicated. The result of the analysis shows a bell would not operate if it did not have "poor on-off control."

9. *The use of television in mathematics education*, by Professor P. S. Jones, University of Michigan. (By invitation).

There have been more than twenty-six television programs or series with mathematical content presented over stations in the United States. Most of these have been of a semi-popular "cultural" nature, designed to interest viewers in mathematics and to inform them about its role and importance. However, there have been several programs designed to teach particular mathematical topics (the slide rule, high school algebra, nomography, the teaching of arithmetic, measurement), and more programs sponsored by public schools showing the content and methods of teaching for public relations purposes.

Television offers opportunities for making individual skilled speakers or teachers and special equipment available to a large audience and hence can do much to increase interest in and appreciation of mathematics and to actually enrich the teaching of mathematics for school children as well as for adult viewers.

MARGARET F. WILLERDING, *Secretary*

## THE APRIL MEETING OF THE OHIO SECTION

The thirty-ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 23, 1955. Professor W. R. Transue, Chairman of the Section, presided at the morning and afternoon sessions.

There were 111 persons registered in attendance including the following 89 members of the Association:

Grace L. Abhau, J. E. Adney, Jr., C. E. Amos, P. R. Annear, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Jack Belzer, J. H. Blau, G. M. Bloom, Foster Brooks, O. E. Brown, Emalou Brumfield, L. E. Bush, F. C. Calabrese, C. D. Calhoon, V. B. Caris, Dorothy I. Carpenter, A. B. Carson, R. A. Clark, E. H. Clarke, G. M. Clough, C. C. Crell, Wayne Dancer, H. F. Davis II, R. W. Emmert, W. S. Ericksen, P. L. Evans, Beverly R. Ferner, H. E. Fettis, W. T. Fishback, M. P. Fobes, G. R. Glabe, H. W. Godderz, R. P. Gosselin, L. J. Green, Arnold Grudin, Marshall Hall, Jr., H. G. Harp, H. L. Harter, R. G. Helsel, Raymond Huck, R. Y. Iwanchuk, S. J. Jasper, M. L. Johnson, Margaret E. Jones, J. W. Kaiser, J. S. Klein, L. C. Knight, Jr., D. M. Krabill, F. A. Kros, P. E. Lauderbach, Nathan Lazar, E. B. Leach, H. D. Lipsich, L. L. Lowenstein, H. R. Mathias, S. W. McCuskey, E. J. Mickle, L. H. Miller, Knox Millsaps, C. C. Morris, O. M. Nikodym, Emma J. Olson, H. S. Pollard, Gustave Rabson, R. F. Reeves, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, S. A. Rowland, H. J. Ryser, B. L. Schwartz, Nancy M. Scribano, R. L. Shively, E. T. Stapleford, R. R. Stoll, Ralph E. Thomas, H. E. Tinnappel, H. S. Toney, W. R. Transue, E. P. Vance, E. H. Wang, Marion D. Wetzel, F. B. Wiley, G. P. Williams, C. O. Williamson, Alberta Wolfe, B. J. Yozwiak.

The following officers were elected for the coming year: Chairman, Professor R. R. Stoll, Oberlin College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Dr. W. E. Deskins, Ohio State University; Program Committee, Chairman, Professor Wade Ellis, Oberlin College; Professor H. D. Lipsich, University of Cincinnati; Dr. E. B. Leach, Case Institute of Technology.

The following papers were presented:

1. *Approaches to measure and integration*, by Professor W. R. Transue, Kenyon College. (Chairman's address).

A comparative discussion is given of several diverse methods of development of the theory of measure and integration. Starting with a consideration of some properties of measurable and integrable sets and functions, and those of measure and integral, there follows a discussion of the use of these properties in the introduction of definitions of these entities. Using Lebesgue measure and integral as an example, an outline is given of 1) the classical development, 2) that based on the Lusin theorem and in use by the Tonelli school, and 3) the procedure of Bourbaki.

2. *The superiority of the trapezoidal rule for a certain class of definite integrals*, by Mr. H. E. Fettis, Wright Air Development Center.

By means of Poisson's summation formula, it can be shown that for certain integrands the error in applying the trapezoidal rule vanishes exponentially as the interval of integration tends to zero. For such functions, the trapezoidal rule can be shown to be superior to Simpson's Rule and other more complicated quadrature formulas. Examples are given of functions for which the magnitude of error can be predicted in advance in terms of the interval of integration.

3. *Note on isosceles orthogonality*. By Professor H. F. Davis, II, Miami University.

Following R. C. James, we say that vectors $v$ and $w$ are isosceles orthogonal if the length of $v+w$ equals the length of $v-w$. Then *a normed linear space is euclidean if and only if orthogonal complements are always subspaces*. A simple direct proof of this theorem is given. The author's interest in this topic arose in connection with directing a master's thesis at Miami University. He feels that original work exploring such concepts is more valuable to the student than a purely expository thesis would be.

4. *Moments of non-integral order*, by Dr. H. L. Harter, Aeronautical Research Laboratory, Wright Air Development Center.

Statisticians attach little meaning to moments of non-integral order and have, until recently, given them little thought. However, certain considerations in the theory of turbulence have led to expressions involving such moments, especially those of order one-half. Moments of order one-half have real values only for distribution functions which are identically zero for all negative values of the variable, and hence when taken about the mean they are always imaginary. Moments about the origin have been computed for several well-known distributions which are restricted to non-negative values, and the results are given in this paper. Absolute moments about the mean are also considered.

5. *Functions of several variables with preassigned derivatives*, by Dr. E. B. Leach, Case Institute of Technology.

The existence of a function of one real variable whose derivatives of all orders have given values at a point has been shown in a simple way by A. Rosenthal. This note makes the following extension: Let $M$ be a $C^\infty$ manifold of dimension $m$, regularly imbedded in $R^n$ (where $m<n$). Then there is a $C^\infty$ function $f$ defined in $R^n$, whose derivatives of all order have given traces on $M$, provided the traces satisfy the "strip conditions" imposed by the chain rule for differentiating a composite function.

6. *Characterizations of determinant functions*, by Professor R. R. Stoll, Oberlin College. (By invitation).

Professor Stoll discussed sets of axioms for determinant functions devised by Schreier and Sperner, Menger and Kozin, Hensel, and Stephanos. Those of Schreier and Sperner were motivated by the following problem: To devise a function on $n$th order matrices which vanishes at $A$ if and only if the homogeneous system of linear equations $AX=0$ has a nontrivial solution. The suggestion was made that one of these approaches might be more stimulating to undergraduates, and more in keeping with current emphasis than the traditional presentation.

7. *A new frontier: photogrammetry*, by Professor H. S. Toney, Wright Air Development Center and University of Dayton.

The field of photogrammetry offers a challenge to mathematicians to use applied mathematical methods to solve problems of loci and distance determinations needed for geodetic survey and airborne reconnaissance. Photogrammetry (only about 100 years old) is defined as the science of obtaining reliable measurements by means of photography (terrestrial or aerial). This introductory paper contains a brief history and a survey of problems in this field which should interest mathematicians.

8. *Social choice and impossibility theorems*, by Professor J. H. Blau, Antioch College.

Arrow's *Social Choice and Individual Values* has been widely quoted and has stimulated further research. In this note it is pointed out that the principal theorems are false. An *ordering* of a set of alternatives is a reflexive, connected and transitive relation on this set. A *social welfare function* (SWF) is a method of combining individual orderings to form a social ordering, subject to several natural conditions. Arrow's Theorems 2 and 3 assert that if certain degrees of variety in individual orderings are permitted, there can be no SWF. An explicit SWF which serves as a counter-example is given here. The error is discussed. On the other hand, if *all* individual orderings are permitted, there can be no SWF. Arrow's methods yield a proof.

9. *A note on Fourier coefficients*, by Professor R. P. Gosselin, Youngstown College.

Let $c(g)$ be the $n$th Fourier (exponential) coefficient of $g(x)$. Let $f(x)$ belong to $L^q$, $q$ an integer $\geqq 2$. Let $c_n(f)$ be positive and decrease with $1/|n|$. By use of Parseval's formula, it is shown that $(c_n(f))^q \leqq A_q c_n(f^q)/(|n|+1)^{q-1}$, where $A_q$ is a constant depending only on $q$. As an application of this inequality, a proof of the following result, due to Hardy and Littlewood (*Journal of the London Math. Soc.*, vol. 6, 1931, pp. 3–9), is obtained: If $f(x)$ belongs to $L^r$, $r \geqq 2$, and $c_n(f)$ decreases, then

$$\sum_{n=-\infty}^{+\infty} |n|^{r-2}(c_n(f))^r \leqq B_r \int^{2\pi} |f(x)|^r dx.$$

10. *The tensor form of the equations of hydrodynamics*, by Mr. W. H. Lane, Wright Air Development Center, introduced by the Secretary.

The tensor forms of the general energy equation and of the Navier-Stokes equations of motion for a viscous incompressible fluid are considered. The special case for spherical coordinates in which the velocity and temperature fields are assumed to be inversely proportional to the power of a radial vector is developed, and the resulting class of exact solutions is discussed.

11. *On ideals in the ring of linear multidifferential polynomials*, by Mr. Frank Levin, University of Cincinnati, introduced by Professor H. D. Lipsich.

The ring of linear multidifferential polynomials is a noncommutative ring of polynomials in several indeterminates. This ring is not a principal ideal ring, and, therefore, the results are stated ideal-theoretically. A basis of an ideal of multidifferential polynomials which corresponds to the basis of an ideal in the principal ideal ring of ordinary differential polynomials is the *canonical basis* of the ideal. With this basis one is able to provide an upper bound for the length of a basis of the ideal and to give a necessary and sufficient condition for solvability of multidifferential equations.

FOSTER BROOKS, *Secretary*

## THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-eighth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on Friday afternoon and evening and Saturday forenoon, April 22 and 23, 1955. Professor Nathan Schwid, Chairman of the Section, presided at all three sessions.

There were 62 persons registered for the meeting, including the following 46 members of the Association:

J. W. Ault, G. E. Bardwell, C. F. Barr, B. C. Bellamy, W. E. Briggs, J. R. Britton, R. K. Butz, F. M. Carpenter, Sarvadaman Chowla, E. L. Crow, W. E. Dorgan, F. N. Fisch, H. T. Guard, Leota C. Hayward, Anna S. Henriques, Archie Higdon, J. E. Householder, Sr., P. F. Hultquist, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, R. B. Kriegh, L. J. Lange, E. B. McLeod, Jr., M. L. Madison, W. D. Marsland, Jr., W. E. Mientka, W. K. Nelson, Greta Neubauer, D. O. Patterson, J. W. Querry, O. M. Rasmussen, O. H. Rechard, A. W. Recht, Calvin A. Rogers, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, P. O. Steen, W. J. Thron, E. P. Tovani, V. J. Varineau, W. W. Varner, C. R. Wylie, Jr.

Officers elected at the meeting for 1955–1956 were: Chairman, Professor

C. R. Wylie, Jr., University of Utah; Vice-Chairman, Professor R. R. Gutzman, Colorado School of Mines; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Remarks on the functional equation* $f[f(z)]=e^z-1$, by Professor W. J. Thron, University of Colorado.

If the functional equation has a solution $f(z)$ which is holomorphic in a sufficiently large neighborhood of the point $z=0$ then it can be shown that $f(0)=0$. Using this one can determine a unique formal power series solution $f(z)=\sum c_n z^n$. Let $r$ be the radius of convergence of this series. By means of Picard's theorem, Hadamard's factorization theorem, and a result of Pólya, it is established that $r<\infty$. The statement that $r=0$ but that there exists a solution of the functional equation which is holomorphic for all $z$, not on the negative real axis, concludes the paper.

2. *Note on hemispheric numerical integration of the barotropic model*, by Major J. F. Blackburn, USAF Academy, and W. L. Gates, presented by Major Blackburn, who was introduced by the Secretary.

Under the assumption of frictionless, adiabatic flow in hydrostatic and quasi-geostrophic equilibrium, the barotropic equation was solved numerically for hemispheric flow. The method of solution was similar to the scheme outlined by Charney, Fjørtoft and von Neumann (*Tellus*, November, 1950) applied to a smaller area. The procedure consists of the cyclical calculation of the absolute vorticity advection at time $t$ for each point of a finite difference grid, the solution of the resulting finite difference equations for $\partial z/\partial t$ at each point by a method of relaxation, and the calculation of the heights $z$ at time $t+\Delta t$ using centered differences over a short time interval.

3. *Some simple geometrical properties of the space* $L^2$ by Mr. A. E. Labarre, Jr., University of Wyoming, introduced by the Secretary.

Geometrical interpretations of the Parseval and Riesz-Fischer theorems are given. The space $L^2$ is the direct product of the even functions of $L^2$ and the odd functions of $L^2$. How the operation of differentiation in $L^2$ can be interpreted as an orthogonal transformation is explained.

4. *The number of lattice points in an n-dimensional tetrahedron*, by Professors Sarvadaman Chowla and W. E. Mientka, University of Colorado, presented by Professor Mientka.

Let the $a_i (1 \leqq i \leqq n)$ be positive integers relatively prime in pairs, and $A=\Pi_{i=1}^{n} a_i$. In this paper we find exact expressions (which are polynomials in $\eta/A$ and the $a_i$) for (*i*) the number of solutions in non-negative integers $x_i$ of $\sum_{i=1}^{n} a_i x_i=\eta$ whenever $\eta \equiv 0 \pmod{A}$, (ii) the number of lattice points in the tetrahedron bounded by the planes $\sum_{i=1}^{n} a_i x_i=\eta$ $(x_i \geqq 0)$ again provided $\eta \equiv 0 \pmod{A}$.

5. *A possible measure of asymmetry in a line*, by Professor Calvin A. Rogers, Colorado Agricultural and Mechanical College.

Eight requirements were set up, formally expressing intuitive convictions about the asymmetry in the $x$-axis of two points $P_1$ and $P_2$ with same abscissa and ordinates, $y_1$ and $y_2$. From these, it was deduced that one of the simplest functions satisfying all requirements was the fraction $(y_1+y_2)^2/(y_1^2+y_2^2)$.

6. *Remarks on the distribution of primes*, by Professors A. J. Kempner and Sarvadaman Chowla, University of Colorado, presented by Professor Kempner.

From the extended Prime Number Theorem two formulae are derived:

$$\frac{\pi(\delta x)-\pi(\gamma x)}{\pi(\beta x)-\pi(\alpha x)}=\frac{\delta-\gamma}{\beta-\alpha}\cdot\left\{1+\mathcal{C}(\alpha,\beta,\gamma,\delta)\cdot\frac{1}{\log x}\right\}+o\left(\frac{1}{\log x}\right)$$

and

$$[\pi(\delta x)-\pi(\gamma x)]-[\pi(\beta x)-\pi(\alpha x)]=[(\delta-\gamma)-(\beta-\alpha)]\cdot\frac{x}{\log x}+\mathcal{C}'(\alpha,\beta,\gamma,\delta)\cdot\frac{x}{\log^2 x}+o\left(\frac{x}{\log^2 x}\right).$$

Specialization of $\alpha, \beta, \gamma, \delta$ leads to results concerning the distribution of primes.

7. *Knots and quadratic forms*, by Professor K. A. Hirsch, University of London and University of Colorado. (By invitation).

The speaker discussed certain topological properties of knots by considering the invariants of related quadratic forms.

8. *The power series coefficients of L-series*, by Dr. W. E. Briggs, University of Colorado.

Consider $L_k(s)=\sum_1^\infty \chi(n)n^{-s}$ where $\chi$ is a real non-principal character mod $k$. This series can be presented by the power series $\sum_1^\infty L^{(r)}(1)(s-1)^r/r!$. These coefficients can be determined by evaluating the $r$-th derivative of the defining series to obtain

$$L^{(r)}(1)=(-1)^r\sum_1^k \chi(l)\gamma_{r,k,l}, \quad \text{where} \quad \gamma_{r,k,l}=\lim_{x\to\infty}\left[\sum_{n\leqq x,\, n\equiv l(k)}\frac{\log^r n}{n}-\frac{\log^{r+1}x}{k(r+1)}\right].$$

Similarly the power series coefficients of the zeta function can be determined by considering

$$h(s)=\zeta(s)-\frac{s}{s-1}=s\int_1^\infty\frac{[x]-x}{x^{s+1}}dx=\sum_0^\infty\frac{h^{(r)}(1)}{r!}(s-1)^r$$

and evaluating the expression obtained by differentiating the integral $r$ times. This gives $h^{(r)}(1)=(-1)^r\gamma_r$ for $r>0$ and $h(1)=\gamma_0-1$ where $\gamma_r=\gamma_{r,1,0}$.

9. *A problem in interpolation*, by M. L. J. Lange, University of Colorado.

Given a polynomial of degree $n$ with coefficients in a field $K$, $f(x)=(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_k)^{m_k}$, and with the $\alpha_i$ in the root field $K'$ of $f(x)$, and given a polynomial $g(z)$ of arbitrary degree with coefficients in $K'$, the problem is to find a polynomial $h(x)$ of degree $\leqq n-1$ with coefficients in an extension field $K''$ such that $F(x)=g(h(x))-x$ is divisible by $f(x)$. The author showed that an $h(x)$ with the required properties exists if and only if for all $\alpha_i$ with $m_i>1$ the equation $g(z)-\alpha_i=0$ has at least one simple root. He also gave a method for actually constructing such a polynomial $h(x)$.

10. *An outline of the mathematics curriculum and schedule at the USAF Academy*, by Colonel Archie Higdon, USAF Academy.

The United States Air Force Academy mathematics curriculum consists of courses in college algebra, plane and spherical trigonometry, analytical geometry, differential and integral calculus, applied calculus, and elementary differential equations for a total of 21 quarter hours.

Students will be sectioned according to demonstrated ability in mathematics with an average of 12.5 students per section. They will be graded almost every day and daily preparation is mandatory. The top sections will cover some advanced topics in each course not required for those with less aptitude for mathematics. These top sections will contain many students who would be admitted with advanced standing in civilian schools. All students are required to complete all four years at the U. S. Air Force Academy regardless of previous college training.

11. *A new Poisson equation analog computer*, by Mr. W. W. Varner, University of Colorado.

A new computer has been completed at the University for the very rapid solution of the general second order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+ku+F(x, y, u)=0$$

with appropriate boundaries. It also solves the Poisson equation in three dimensions with a grid of 960 nodes or mesh points which can be arranged very easily and quickly into Cartesian, cylindrical, spherical, and other coordinate systems. It can handle very complicated boundaries, source and sink conditions, and transients.

12. *A review of the* 1954 *Oregon Summer Conference*, by Professor F. M. Carpenter, Colorado School of Mines.

F. M. CARPENTER, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA, South Dakota State College, Brookings, October 15, 1955.

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA

OHIO, April, 1956.

OKLAHOMA, Oklahoma City University, October 28, 1955.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

ROCKY MOUNTAIN

SOUTHEASTERN, UNIVERSITY OF GEORGIA, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.
No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
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MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

# CONTRIBUTIONS TO GEOMETRY

*The Fourth*
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPER

This publication has been financed by the Arnold Buffum Chace Fund
of the Mathematical Association of America

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY
Volume 62 Number 7
August–September, 1955

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## EDITORIAL NOTE

This number of the *Slaught Papers* consists of a collection of articles in the field of geometry which were submitted independently to the MONTHLY. Because of the close connection among several of these it appeared desirable to publish them together in a single issue. Since enough space was not available in a regular issue of the MONTHLY for this purpose, they are being published together in this *Slaught Paper*.

The first article by Bruck is a summary of the present status of work on Euclidean plane geometry since the time of Hilbert. It discusses the effects of modifications of the axioms, and relates the various geometries so obtained to appropriate algebraic systems. The succeeding articles by Hall, Ryser, and Wesson elaborate on special aspects of this theme.

The remaining articles are less directly related to this central theme, but illustrate other features of recent work in geometry. Stockton writes within the framework of classical projective geometry. Rosenfeld presents an axiomatic treatment of analytic trilinear geometry. Court gives new results in the classical field of "modern geometry." Finally Gans discusses the foundations of Elliptic Geometry.

It is hoped that the simultaneous publication of these papers will serve to arouse the interest of mathematicians in the currently under-emphasized field of geometry.

# RECENT ADVANCES IN THE FOUNDATIONS OF EUCLIDEAN PLANE GEOMETRY*

R. H. BRUCK, University of Wisconsin

**1. Introduction.** A program of axiomatizing Euclidean plane geometry in a manner consistent with present standards of rigour was beautifully carried out by Hilbert [1].§ Upon reading Hilbert's book in its entirety one sees what is not at first evident—that Hilbert is alive to the interesting questions which arise when some of his geometric axioms are dropped or modified. And many mathematicians, both before and since the appearance of Hilbert's book, have investigated such problems.

Unfortunately for the wide audience whose interest in geometry was awakened in high school or university, answers to the deeper questions of the sort I have in mind, if they have been given at all, require a long excursion into abstract algebra. I know of no remedy for this situation. What can be done—or, at any rate, what the present paper attempts to do—is to give a pictorial account of some of the geometric axioms, a simple explanation of the algebraic problems which these pose and a brief account (with references rather than proofs) of the answers.

Think of this paper as an excursion from wherever you are towards a town named Cayley Numbers. As we saunter along, we pass by Planar Ternary Rings, Veblen-Wedderburn Systems, Division Rings with the Right Inverse Property, Right Alternative Division Rings, Alternative Division Rings—and down the hill we see Cayley Numbers.—Strange names they have for towns in these parts, but I believe you'll enjoy the scenery.

**2. The axioms of incidence.** We shall use geometric language quite informally. In particular we shall assume that everyone feels at home with phrases such as "point is on line," "line is through point" and with adjectives such as "collinear," "concurrent," "parallel." Such carelessness is a little dangerous (since the language may have unintended connotations) but saves a great many words.

A Euclidean (or affine) plane $\pi$ is a system of undefined objects, called points and lines, subject to the following *axioms of incidence:*

(i) *If $P$, $Q$ are distinct points of $\pi$, there is one and only one line, $PQ$, of $\pi$, through both of $P$ and $Q$.*

(ii) *If $a$, $b$ are distinct lines of $\pi$, there is at most one (and there may be no) point of $\pi$ on both of $a$ and $b$.*

* This paper had its origin in 1950 in my seminar at the University of Wisconsin. It has been presented since then by my students or myself in various forms at several universities. The present account was delivered by invitation to the Iowa Section of this Association in Ames, Iowa, April 30, 1954.

§ The book [1a] is included because it is in English and indicates the spirit of Hilbert's later approach but all specific references [1] are to [1b].

(iii) *If the point $P$ of $\pi$ is not on the line $a$ of $\pi$, there is exactly one line of $\pi$ which passes through $P$ and is parallel to $a$.*

(iv) *There is at least one set, $A, B, C, D$, of four distinct points of $\pi$, no three of which are collinear.*

The axioms of incidence require so little of a Euclidean plane that very few theorems have been proved. Indeed, the main "theorem"—I like to call it Hall's Theorem—might be stated as follows: Any damn thing can happen. As a relatively respectable example of this we may note that the following system satisfies the axioms: $\pi$ consists of four distinct points, $A, B, C, D$, and of six distinct lines, namely the following three pairs of parallels: $AB, CD$; $AC, BD$; $AD, BC$. Each line contains exactly two distinct points and each point lies on exactly three distinct lines.

Even so, the axioms of incidence do ensure a certain amount of regularity. Suppose $a$ is a line of $\pi$ and $P$ is a point of $\pi$ not on $a$. By (iii), there is exactly one line through $P$, say $b$, which does not meet $a$. The remaining lines through $P$ are in one-to-one correspondence with the points of $a$, for,* by (i), (ii), (iii), each of these lines is a line $PQ$ for exactly one point $Q$ of $a$, and conversely. Hence we can say that there is "one more" line through $P$ than there are points of $a$. Consequently, every two lines not through $P$ have the same number of points. Then, considering the points $A, B, C, D$ of (iv), we see that every line not through $A$ has the same number of points as $BC$ or $BD$ or $CD$, and similarly for $B, C, D$. Thus, if some line of $\pi$ has $n$ points (where $n$ is a positive integer or a transfinite cardinal) then every line has $n$ points and every point lies on $n+1$ lines.

We can add: if $a, b, c$ are distinct lines and if $a$ is parallel to $b$ and $b$ parallel to $c$, then $a$ is parallel to $c$—else the common point of $a, c$ would have through it two parallels to $b$, namely $a, c$. Now consider two lines $a, d$, meeting in a point $P$. Each of the $n-1$ points of $d$, other than $P$, determines a unique parallel to $a$, and conversely. Hence the *parallel class* of $a$, consisting of $a$ and the lines parallel to $a$, contains precisely $n$ lines. Moreover the parallel class of $a$ is the parallel class of each of the lines contained in it.

In a rigorous treatment of Euclidean planes it is necessary at various points to give special consideration to planes with a small number of points on each line. We shall be content to ignore this difficulty entirely.

**3. Hall's planar ternary rings.** How can we introduce coordinates into a Euclidean plane $\pi$ subject only to the axioms of incidence? We cannot talk of rectangular axes—since we have no notion of angle aside from the special case of parallel lines. We cannot talk of lengths—we have no notion of distance. We cannot talk of the slope of a line—but, on the contrary, we shall do just that in a moment.

---

* A few simple diagrams, which we feel compelled to omit, make the following remarks quite obvious.

The method of Marshall Hall [2] seems as good as can be expected. We select an arbitrary point $O$ of $\pi$ and three distinct lines§ through $O$ which we call the $x$-axis, the $y$-axis and the unit line (Fig. 1). On the unit line we select any point $I$ (the unit point) distinct from $O$. The line through $I$ parallel to the $y$-axis we call the slope line.

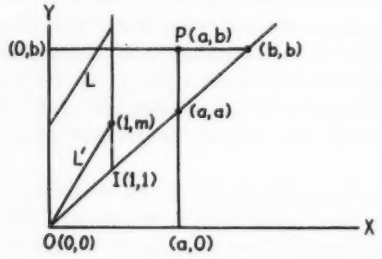


FIG. 1

Now we choose an arbitrary set $R$ of elements or "labels" subject to two restrictions: (a) Among the elements of $R$ are two distinct elements 0, 1. (b) The elements of $R$ can be put into one-to-one correspondence with the points of the unit line $OI$ (and hence with the points of any line of $\pi$). To each point of the unit line $OI$ we assign a pair of coordinates $(x, y)$ with $x=y$, where $x=y$ is an element of $R$. In particular we give the origin $O$ the coordinates (0, 0) and the unit point $I$ the coordinates (1, 1). We arrange that each point of $OI$ has coordinates $(a, a)$ for a uniquely determined element $a$ of $R$ and that, conversely, there is a unique point on $OI$ with coordinates $(b, b)$ for each element $b$ of $R$.— Despite all this arbitrariness in assigning coordinates to the points of $OI$, we use the coordinates in such a way that any two ways of assigning them would be equally good or bad.

Next consider any point $P$ of $\pi$. The line through $P$ in the parallel class of the $y$-axis meets $OI$ in a unique point; say the point with coordinates $(a, a)$. And the line through $P$ in the parallel class of the $x$-axis meets $OI$ in a unique point, say $(b, b)$. Then we assign to $P$ the coordinates $(a, b)$ (see Fig. 1). In particular, points of the $x$-axis have coordinates of form $(x, 0)$, points of the $y$-axis have the form $(0, y)$, and the four points $(0, 0)$, $(x, 0)$, $(0, y)$, $(x, y)$ form the vertices of a parallelogram.

The line through the point $(a, b)$ in the parallel class of the $y$-axis will naturally have the equation $x=a$. Similarly, it is clear what we mean by the line with equation $y=b$. We have yet to assign equations to the other lines, except that the unit line $OI$ should certainly have the equation $y=x$.

Consider any line $L$. There is a unique line $L'$ in the parallel class of $L$ which

§ Note that, in Figure 1, $OX$, $OY$ are perpendicular and angle $XOY$ is bisected by the unit line $OI$. This is meaningless but somehow comforting.

passes through the origin $O$. If $L'$ is the $y$-axis, we assign no slope to $L$ or $L'$. Otherwise, $L'$ must meet the slope line $x=1$ in a unique point, say the point $(1, m)$; in this case we assign to $L$ (and $L'$) the slope $m$ (see Fig. 1). In particular, every line $y=b$ has slope 0, and every line in the parallel class of the unit line $OI$ has slope 1.

At this stage every point of $\pi$ has a unique pair of coordinates and every line, except for the lines $x=a$, has a unique slope. Now consider a line $L$ which intersects the $y$-axis in the "$y$-intercept" $(0, b)$. This line $L$ has a unique slope $m$. We should *like* to be able to say that $L$ has equation $y=xm+b$; but, at the present stage,* at least, such an equation is meaningless. Instead, we use the plane $\pi$ and the coordinate system which we have set up to define a *ternary operation* (or function) $F$ on the elements of the label set $R$, in such a manner that, for each ordered triple $a$, $m$, $b$ of elements of $R$, $F(a, m, b)$ is an element, say $d$, of $R$. This is done as follows: for any $m$, $b$, consider the line $L$ of slope $m$ through the $y$-intercept $(0, b)$. The line $x=a$ meets the line $L$ in a unique point $P$ whose coordinates are $(a, d)$ for some definite element $d$ of $R$. Then we define $F(a, m, b)=d$.

In view of the definition of $F$ it is tautological to say (see Fig. 2) that the equation of the line with slope $m$, $y$-intercept $(0, b)$ is $y=F(x, m, b)$. As a particular case, the unit line $OI$ has slope 1 and $y$-intercept $(0, 0)$. The equation of $OI$ is surely $y=x$ and yet it is also $y=F(x, 1, 0)$. Consequently, $F(x, 1, 0)=x$ for every $x$ in $R$.—There are other facts about $F$ which arise, like this, directly from the definitions, and there are deeper facts which come by insisting upon the full import of the axioms of incidence.

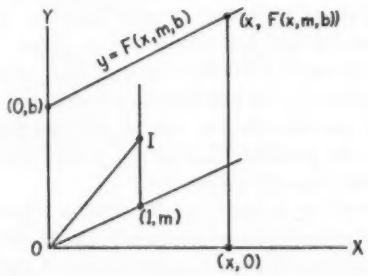


FIG. 2

The system $(R, F)$, consisting of the label set $R$ and the ternary operation $F$, is known as a *planar ternary ring*. Such systems are of course useless unless we have some way of singling out planar ternary rings from among all systems with ternary operations—for a suitable set of postulates see Appendix I. And even

* After we impose the vector axiom, equations of lines will indeed take this familiar form. (See §§ 6, 7, 8.)

then they are relatively useless until we find some way of handling them more easily than the Euclidean planes themselves.

**4. Addition.** For any planar ternary ring the operation of addition $(+)$ is defined equationally by

$$a + b = F(a, 1, b), \qquad \text{all } a, b \text{ in } R. \tag{4.1}$$

The algebraic consequence of (4.1) is this: *the equation of the line with slope 1 and y-intercept* $(0, b)$ *can now be written* $y = x + b$.

The geometric counterpart of (4.1) is the notion of addition of points on the unit line $OI$. Every ordered pair $A$, $B$ of points $A = (a, a)$, $B = (b, b)$ of $OI$ uniquely determines a sum-point $S = (a+b, a+b)$ of $OI$ as follows (see Fig. 3):

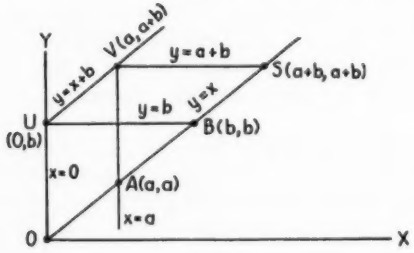


FIG. 3

the line $(y = b)$ of slope 0 through $B = (b, b)$ meets the $y$-axis in point $U = (0, b)$. The line $(y = x + b)$ of slope 1 through $U$ and that line $(x = a)$ through $A$ which is in the parallel class of the $y$-axis meet in a point $V = (a, a+b)$. The line $(y = a + b)$ of slope 0 through $V$ meets the unit line $(y = x)$ in $S = (a+b, a+b)$. This geometric operation can be explained, without reference to coordinates or axes, in terms of three parallel classes: the lines in the parallel class of the $x$-axis (slope 0); the lines in the parallel class of the $y$-axis (no slope); the lines in the parallel class of the unit line $OI$ (slope 1).

If we write $A + B = S$, it is easily verified from Figure 3 that if any two of $A$, $B$, $S$ are arbitrarily assigned as points of $OI$, the third is uniquely determined by the equation. Moreover $A + 0 = A$ and $0 + A = A$ for every point $A$ of $OI$. This means (see Appendix I) that *the system* $(R, +)$ *is a loop*.

Now let us suppose, temporarily, that the Euclidean plane $\pi$ satisfies all the usual axioms of high-school geometry, so that we can make use of line segments.* In Figure 3, from the parallelogram $AOUV$, $\underline{OA} = \underline{UV}$, and, from the parallelogram $UVBS$, $\underline{UV} = \underline{BS}$. Consequently, $\underline{OA} = \underline{BS}$, so that the sum of the line segments $\underline{OA}$, $\underline{OB}$ is equal to $\underline{OS}$. In this special case, then, the somewhat

* In order to distinguish between the line through the two points $A$, $B$ and the line segment (or vector—see § 6) with initial point $A$, endpoint $B$, we underline the latter. Thus: line $AB$, line segment $\underline{AB}$.

arbitrary equation $A+B=S$ is illuminated by the equation $\underline{OA}+\underline{OB}=\underline{OS}$. The equation (4.1) is thus related to very familiar things indeed. We shall return to this subject in §6.

**5. Multiplication.** For any planar ternary ring $(R, F)$ the operation of multiplication $(\cdot)$ is defined equationally by

$$ab = F(a, b, 0), \qquad \text{all } a, b \text{ in } R. \tag{5.1}$$

The algebraic consequence of (5.1) is this: *the equation of the line through $O=(0, 0)$ with slope $m$ can now be written $y=xm$.*

The geometric counterpart of (5.1) is the notion of multiplication of points on the unit line $OI$. Every ordered pair $A$, $B$ of points $A=(a, a)$, $B=(b, b)$ of $OI$ uniquely determines a product-point $P=(ab, ab)$ of $OI$ as follows (see Fig. 4): The line $(y=b)$ of slope 0 through $B$ meets the slope line $(x=1)$ in a point $U=(1, b)$. The line $OU(y=xb)$ of slope $b$ meets the line $(x=a)$ of no slope through $A$ in a point $V=(a, ab)$. The line $(y=ab)$ of slope 0 through $V$ meets the unit line $(y=x)$ in the point $P=(ab, ab)$. This geometric operation can be explained, without reference to coordinates or axes, in terms of three classes of lines: the parallel class of the $x$-axis (slope 0); the parallel class of the $y$-axis (no slope); the class consisting of all lines through $O$ except the $y$-axis (one line for every slope).

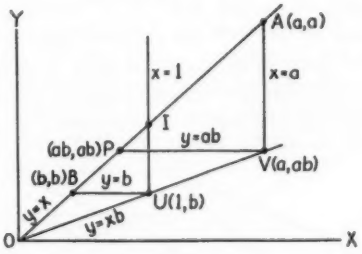


FIG. 4

If we write $A \cdot B = P$, we first note from Figure 4 that $A \cdot 0 = 0 \cdot A = 0$ for every point $A$ of $OI$. Then, if we restrict attention to the set $S$ consisting of the points of $OI$ other than $O$, we find from Figure 4 that if any two of $A$, $B$, $P$ are assigned in $S$, the equation $A \cdot B = P$ uniquely determines the third as a point in $S$. Moreover, $I \cdot A = A \cdot I = A$ for every $A$ in $S$. This means (see Appendix I) that, if $R^*$ denotes the set of elements of $R$ exclusive of 0, *the system $(R^*, \cdot)$ is a loop.*

Again let us assume, temporarily, all the axioms of high school geometry. In Figure 4 we take $A \neq O$, $B \neq O$. From two sets of similar triangles,

$$\underline{OA}:\underline{OI} = \underline{OV}:\underline{OU} = \underline{OP}:\underline{OB},$$

so that

$$\underline{OA}\cdot\underline{OB} = \underline{OI}\cdot\underline{OP}.$$

Consequently, if we take $\underline{OI}$ to have unit length, we can parallel the abstract equation $A\cdot B = P$ with the familiar equation $\underline{OA}\cdot\underline{OB} = \underline{OP}$. (Although Hilbert introduces enough axioms to validate these calculations, we shall not quite do so.)

**6. The vector axiom.** At the end of §4 we temporarily made use of the notion of a line segment. In our usual thinking a line segment $\underline{AB}$ has both an inside and an outside. Such concepts require axioms of order (see Hilbert [1]) which we do not wish to introduce. We shall be content with a very mild notion of a *vector*. For present purposes, a vector $\underline{AB}$ consists merely of an ordered pair of distinct points $A$, $B$: an *initial point* $A$ and an *endpoint* $B$. As before $AB$ denotes the *line* through $A$ and $B$.

We introduce a natural notion of equality of vectors. First of all, any vector is equal to itself: $\underline{AB} = \underline{AB}$. Next, if $AB$, $A'B'$ are distinct lines, the vectors $\underline{AB}$, $\underline{A'B'}$ will be called equal if and only if the lines $AB$, $A'B'$ are parallel and the lines $AA'$, $BB'$ are parallel. (Note that, in this case, if $\underline{AB} = \underline{A'B'}$ then, also, $\underline{BA} = \underline{B'A'}$, $\underline{AA'} = \underline{BB'}$, $\underline{A'A} = \underline{B'B}$.) Next suppose that $AB$, $A'B'$, $A''B''$ are distinct lines and that, according to our definition, $\underline{AB} = \underline{A'B'}$ and $\underline{A'B'} = \underline{A''B''}$. Our notion of equality will be useless unless we can be sure that $\underline{AB} = \underline{A''B''}$. It certainly is true that $AB$, $A''B''$ are parallel. Moreover, if $A$, $A'$, $A''$ are collinear then $B$, $B'$, $B''$ are collinear and the line $AA'' = AA'$ is parallel to the line $BB'' = BB'$; so that, in this case, it is true that $\underline{AB} = \underline{A''B''}$. But if $A$, $A'$, $A''$ are not collinear the desired equality need not* hold. Therefore we force equality by imposing the vector axiom (see Fig. 5):

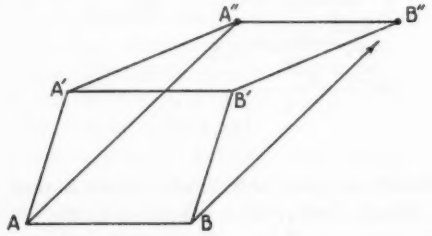


FIG. 5

THE VECTOR AXIOM. *If six distinct points of the Euclidean plane $\pi$ form two triangles $AA'A''$, $BB'B''$, if the lines $AB$, $A'B'$, $A''B''$ are parallel and if the pairs of lines $(AA', BB')$, $(A'A'', B'B'')$ are parallel, then $AA''$, $BB''$ are also parallel.*

* There exist at least two essentially different Euclidean planes of order 9 (9 points on each line) in which the vector axiom fails. One type can be deduced from Hall [2] and another will be found in Carmichael [3].

The vector axiom is a special case of the axiom of Desargues and is one of the axioms used by Hilbert [1].

The vector axiom can be reached in still another way. According to the usual definition, the sum of two vectors $\underline{AA'}$, $\underline{A'A''}$ is the vector $\underline{AA''}$. If $\underline{BB'}=\underline{AA'}$ and $\underline{B'B''}=\underline{A'A''}$, we should like the sum $\underline{BB'}+\underline{B'B''}=\underline{BB''}$ to be equal to $\underline{AA''}$. If $A$, $A'$, $A''$ are collinear, this is automatic, but in the case of Figure 5 we require the vector axiom.

Before we can go on, there is still one more aspect of vector equality which requires consideration. Suppose we have $\underline{AB}=\underline{CD}$ and $\underline{CD}=\underline{EF}$, where the lines $AB$, $CD$ are distinct but $AB$, $EF$ are identical. We would like to say that $\underline{AB}=\underline{EF}$, but in doing so we are in danger of serious trouble. For suppose that also $\underline{AB}=\underline{C'D'}$ and $\underline{C'D'}=\underline{EG}$; we need to be able to assert that $F=G$. Luckily no new axiom is needed. We will indicate how this is so by examining the case that the lines $AB$, $CD$, $C'D'$ are distinct. In this case, since $\underline{C'D'}=\underline{AB}$ and $\underline{AB}=\underline{CD}$, the vector axiom gives $\underline{C'D'}=\underline{CD}$. Then, since $\underline{C'D'}=\underline{CD}$ and $\underline{CD}=\underline{EF}$, the vector axiom gives $\underline{C'D'}=\underline{EF}$. However, $\underline{C'D'}=\underline{EG}$. Therefore the line through $D'$ parallel to $C'E$ meets the line $ABEF$ in $F$ and in $G$, and consequently $F=G$.—For the case that $CD$, $C'D'$ are the same line we simply introduce a new line $C''D''$ and argue as before.

At this stage, assuming the vector axiom, we can be confident that equality of vectors satisfies the usual reflexive, symmetric and transitive laws. There only remains to introduce zero vectors $\underline{AA}$ (with the same initial and final points) and to define $\underline{AA}=\underline{BB}$ for all points $A$, $B$; or, more conveniently, $\underline{AA}=0$ for all points $A$. We also give a symmetric definition of vector addition as follows: If $\underline{AB}$, $\underline{CD}$ are any two vectors, choose any point $P$, determine $Q$ so that $\underline{PQ}=\underline{AB}$ and $R$ so that $\underline{QR}=\underline{CD}$, and call $\underline{PR}$ the sum of the ordered pair of vectors $\underline{AB}$, $\underline{CD}$. It is easy to check that (in the sense of vector equality) the sum is independent of the point $P$.

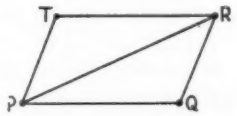


FIG. 6

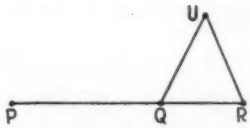


FIG. 7

Now the associative law of vector addition is evident, since $(\underline{PQ}+\underline{QR})+\underline{RS}=\underline{PR}+\underline{RS}=\underline{PS}$ and $\underline{PQ}+(\underline{QR}+\underline{RS})=\underline{PQ}+\underline{QS}=\underline{PS}$. The commutative law of addition is equally evident for vectors $\underline{PQ}$, $\underline{QR}$ if $P$, $Q$, $R$ are not collinear, since $\underline{PQ}+\underline{QR}=\underline{PR}$ while (see Fig. 6) $\underline{QR}+\underline{PQ}=\underline{PT}+\underline{TR}=\underline{PR}$. On the other hand, if $P$, $Q$, $R$ are collinear, a simple device (see Fig. 7) allows us to use the non-collinear case along with associativity: $\underline{QR}+\underline{PQ}=(\underline{QU}+\underline{UR})+\underline{PQ}=\underline{QU}+(\underline{UR}+\underline{PQ})=\underline{QU}+(\underline{PQ}+\underline{UR})=(\underline{QU}+\underline{PQ})+\underline{UR}=(\underline{PQ}+\underline{QU})+\underline{UR}=\underline{PQ}+(\underline{QU}+\underline{UR})=\underline{PQ}+\underline{QR}$. Consequently, all the usual laws of vector addition are satisfied.

Now we are ready to consider the additive system $(R, +)$ defined by (4.1). More specifically, we consider Figure 3. In Figure 3, $\underline{OA} = \underline{UV}$ and $\underline{UV} = \underline{BS}$, so $\underline{OA} = \underline{BS}$. Therefore $\underline{OS} = \underline{OB} + \underline{BS} = \underline{OB} + \underline{OA} = \underline{OA} + \underline{OB}$. And, inasmuch as $S = (a+b,\ a+b)$, $A = (a,\ a)$, $B = (b,\ b)$, we can assert the following (see Appendix I):

*In the presence of the vector axiom, the system $(R, +)$ is an abelian group isomorphic to the additive group of vectors.*

**7. Linearity.** If the vector axiom had not already been thrust upon us in connection with equality of vectors, we could urge another reason for its adoption. Namely, we would like every planar ternary ring $(R, F)$ of the Euclidean plane $\pi$ to have the property of *linearity* embodied in

$$F(a, b, c) = ab + c \qquad \text{all } a, b, c, \text{ in } R. \tag{7.1}$$

where the addition and multiplication on the right hand side are as defined in (4.1), (5.1). The algebraic consequence of (7.1) is this: *the equation of the line with slope $m$, $y$-intercept $(0, b)$ can be written $y = xm + b$.*

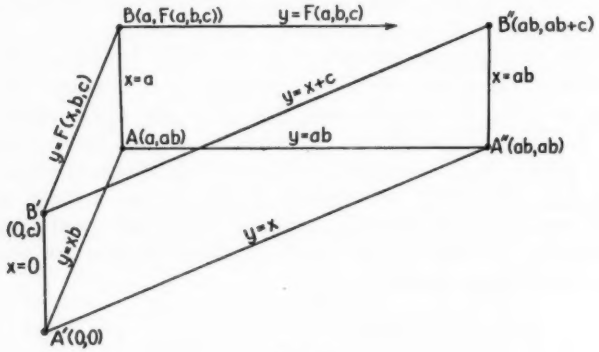


FIG. 8

In examining (7.1) we take $a \neq 0$, $c \neq 0$, $b \neq 0, 1$, since otherwise (7.1) holds trivially. Consider Figure 8, which emphasizes the essential nature of (7.1) by omitting irrelevant points (such as the unit point $I$) and lines (such as the $x$-axis). The $y$-axis appears as the line $x = 0$, and the unit line $OI$ as the line $y = x$. (7.1) will hold if and only if the line $y = F(a, b, c)$ passes through the point $(ab,\ ab+c)$; that is (in Fig. 8) if and only if the lines $AA''$, $BB''$ are parallel. Hence, by comparison of Figure 8 with Figure 5, we see that, in the presence of the vector axiom, every planar ternary ring of $\pi$ is linear. Now assume conversely that every planar ternary ring of $\pi$ is linear, and consider Figure 5. With a little care we can construct a coordinate system in which the points and lines of Figure 5 have coordinates and equations of the forms indicated in Figure 8; then, by linearity, we can deduce that $BB''$ is parallel to $AA''$. To sum up:

*A necessary and sufficient condition that every planar ternary ring of the Euclidean plane $\pi$ be linear is that $\pi$ satisfy the vector axiom.*

**8. Veblen-Wedderburn systems.** The vector axiom has still another consequence, namely the right distributive law

$$(a+b)c = ac + bc, \qquad \text{all } a, b, c \text{ in } R, \tag{8.1}$$

of multiplication with respect to addition. We first note that (8.1) holds trivially if any one of $a$, $b$, $c$ is zero or if $c=1$. Excluding these cases, consider Figure 9 below. The line $OA'(y=xc)$ meets the line $C'D'(x=a+b)$ in the point

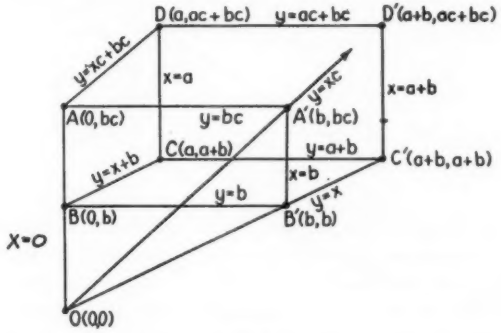


FIG. 9

$D''=(a+b, (a+b)c)$. We want to prove that $D''$ coincides with $D'=(a+b, ac+bc)$. By use of vectors, $\underline{A'D'}=\underline{A'B'}+\underline{B'C'}+\underline{C'D'}$. By vector equality, $\underline{A'B'}=\underline{AB}$, $\underline{B'C'}=\underline{BC}$, $\underline{C'D'}=\underline{CD}$. Therefore $\underline{A'D'}=\underline{AB}+\underline{BC}+\underline{CD}=\underline{AD}$. In particular, then, the line $A'D'$ has the slope of $AD$, namely $c$. However, $OA'$ has slope $c$, so $O$, $A'$, $D'$ are collinear. That is, $OA'$ meets $C'D'$ in $D'$, proving that $D''=D'$. This completes the proof of (8.1).

To sum up, in the presence of the vector axiom each planar ternary ring $(R, F)$ is linear, the additive system is an abelian group and the right distributive law (8.1) holds. A planar ternary ring with these properties is known as a Veblen-Wedderburn system, after O. Veblen and J. H. M. Wedderburn, who first studied* such systems in 1907. (For a complete set of postulates, see Appendix II.) It can be shown, conversely, that if one planar ternary ring of $\pi$ is a Veblen-Wedderburn system then the vector axiom holds. Therefore:

THEOREM 1. *If the vector axiom holds in a Euclidean plane $\pi$, then every planar ternary ring of $\pi$ is a Veblen-Wedderburn system. Conversely, if any one ternary ring of $\pi$ is a Veblen-Wedderburn system, then the vector axiom holds in $\pi$.*

The theory of Veblen-Wedderburn systems is still relatively undeveloped—

* See [4]. The discussion in [2] is better suited to present purposes.

though I would hazard a guess that abstract algebra soon will be able to cope with these systems. In the meantime we are much hampered by the lack of the left distributive law, which must be paid for with an additional geometric axiom.

**9. The distributive axiom.** Our next axiom (see Fig. 10) is worded§ for ready comparison with one of Hilbert's:

THE DISTRIBUTIVE AXIOM. *Let seven distinct points of the Euclidean plane consist of two triangles $ABC, A'B'C'$ in perspective from a point $O$, with the pairs $(AB, A'B')$, $(BC, B'C')$ of corresponding sides parallel and with (*) $BC$ parallel to $OA$. Then the third pair of sides, $AC, A'C'$, are also parallel.*

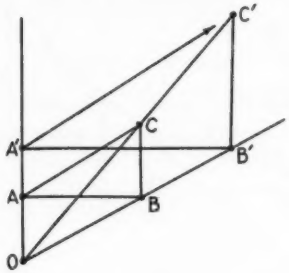


FIG. 10

The corresponding axiom of Hilbert [1] is stronger in that the restrictive hypothesis (*) is omitted. Both of these axioms are special cases of the axiom of Desargues.

Now consider the identity

$$F(a, b, ac) = aF(1, b, c), \qquad \text{all } a, b, c \text{ in } R. \tag{9.1}$$

In the presence of (7.1) (since $1b = b$ for every $b$) (9.1) is equivalent to the left distributive law

$$ab + ac = a(b + c), \qquad \text{all } a, b, c \text{ in } R. \tag{9.2}$$

Therefore we are interested in the geometric axiom which asserts (9.1) for every planar ternary ring $(R, F)$ of $\pi$. Just as we showed that (7.1) was equivalent to the vector axiom, so we can show that (9.1) is equivalent to the distributive axiom. This is indicated by Figure 11 below.

**10. Division rings with the right inverse property.** Recall the common saying: "You get out of anything just what you put into it." Surely one would have to work hard to justify such a statement in mathematics. For example, if we put into the Euclidean plane $\pi$ just enough to ensure that every coordinate ring of $\pi$ is linear (the vector axiom) we get out all the properties of Veblen-

§ It would be neater to say that the three lines $OAA'$, $BC$, $B'C'$ are parallel.

Wedderburn systems, including the right distributive law. When we insist further on the left distributive law (9.2) for every coordinate ring of $\pi$ (that is, on the distributive axiom) the harvest is even more remarkable, as we shall see.

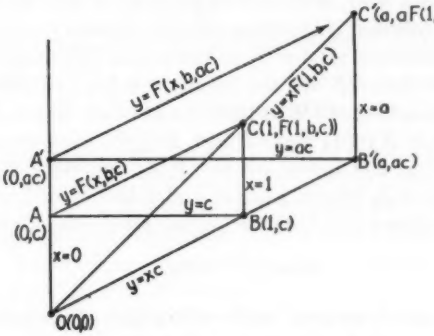


FIG. 11

A Veblen-Wedderburn system which also satisfies the left distributive law is much better known as a *division ring* (with identity element). (For a complete set of postulates see Appendix II.) Thus: *A necessary and sufficient condition that every planar ternary ring of a Euclidean plane $\pi$ be a division ring is that $\pi$ satisfy both the vector axiom and the distributive axiom.* In Theorem 1 it was stated that if any one coordinate ring of a Euclidean plane is a Veblen-Wedderburn system then (the vector axiom holds and) all are. A comparable statement would be false here: there exist Euclidean planes in which some but not all of the coordinate rings are division rings, the rest being merely Veblen-Wedderburn systems. For the correct theorem we need the notion of the right inverse property.

A division ring $R$ (with identity element 1) is said to have the *right inverse property* if each nonzero element $a$ of $R$ has an inverse $a^{-1}$ such that

$$(10.1) \qquad (ba)a^{-1} = b, \qquad \text{all } a, b \text{ in } R,\ a \neq 0.$$

(From (10.1) with $b=1$, $aa^{-1}=1$; thus (10.1) yields the weak associative law $(ba)a^{-1}=b(aa^{-1})$.) The correct theorem is as follows:

THEOREM 2. *The following properties are equivalent for a Euclidean plane $\pi$:*

(i) *$\pi$ satisfies the vector axiom and the distributive axiom.*

(ii) *Every planar ternary ring of $\pi$ is a division ring.*

(iii) *Every planar ternary ring of $\pi$ is a division ring with the right inverse property.*

(iv) *Some planar ternary ring of $\pi$ is a division ring with the right inverse property.*

We have already seen that (i) is equivalent to (ii) and we shall be content

to show now that (i) implies (iii). Assuming (i), we have that every planar ternary ring $(R, F)$ of $\pi$ is a division ring. Consider Figure 12 below. If to Figure 12 we add another triangle $A'B'C'$ in such a manner that the hypotheses of the distributive axiom hold, $A'C'$ will be parallel to $AC$. We may phrase this more conveniently as follows: if the lines $OB$, $OC$ remain fixed, the slope of $AC$ is the same for every choice of $A (A \neq O)$ on the $x$-axis. We take $A = (b, 0)$ for any nonzero $b$. We assume that $OB$ has the fixed slope $1+a$, distinct from 1, 0; so that $a \neq 0, -1$. The equation of $OB$ is then $y = x(1+a)$. Since $AB$ has equation $x = b$, the $y$-coordinate of $B$ is $b(1+a) = b+ba$. $BC$, $OC$ have equations $y = b+ba$, $y = x$ respectively, so that $C = (b+ba, b+ba)$. If $AC$ has slope $m$, the equation of $AC$ is $y = xm+k$ for some $k$, where, since $A$ and $C$ lie on the line, $0 = bm+k$ and $b+ba = (b+ba)m+k = (bm+k)+(ba)m = (ba)m$. Therefore $m$ satisfies

$$(ba)m = b + ba. \tag{10.2}$$

Since $m$ is independent of $b$ we may set $b = a^{-1}$ where $a^{-1}$ is defined by $a^{-1}a = 1$. Then (10.2) yields $m = a^{-1}+1$. Hence, for all $a \neq 0, -1$ and $b \neq 0$, $(ba)(a^{-1}+1) = b+ba$. From this we get the right inverse property (10.1).—Strictly speaking, we must examine (10.1) for the case $b = 0$ and the case $a = -1$, but these give no trouble.

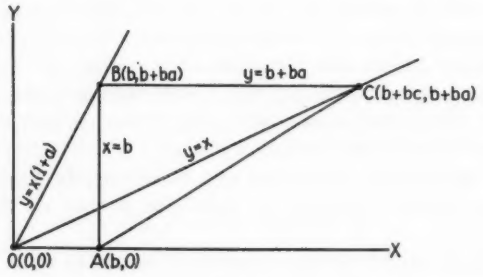


FIG. 12

**11. The algebra begins in earnest.** At this stage it becomes important to give an algebraic characterization of division rings with the right inverse property. The main theorems are as follows:

THEOREM 3. *Every division ring with the right inverse property is an alternative division ring, and conversely.*

THEOREM 4. *Every alternative division ring is either an associative division ring (that is, a field or skew-field) or a Cayley division algebra over its centre.*

We shall discuss these theorems briefly. It may be shown that a division ring $R$ (with identity element 1) has the right inverse property if and only if it satisfies the identity

$$((ab)c)b = a((bc)b), \qquad \text{all } a, b, c \text{ in } R. \tag{11.1}$$

From (11.1) with $c=1$ we derive

$$(ab)b = a(bb), \qquad \text{all } a, b \text{ in } R. \tag{11.2}$$

A ring satisfying (11.2) is called *right alternative;* and it is called *alternative* if it satisfies both (11.2) and

$$b(ba) = (bb)a, \qquad \text{all } a, b \text{ in } R. \tag{11.3}$$

In 1950, Skornyakov [5] proved Theorem 4 for characteristics other than 2, 3. Quite independently, Bruck and Kleinfeld [6] proved Theorem 4 for characteristic not 2 and later Kleinfeld [7], by combining the methods of the first two papers, removed the restriction as to characteristic. Since then, Kleinfeld [8] has given a definitive characterization of simple alternative rings and, incidentally, a new proof of Theorem 4.

In 1951, only a few months after I had become aware of Theorem 2, Skornyakov [9] proved that every right alternative division ring of characteristic not 2 is alternative. In the case of characteristic not 2, (11.2) implies (11.1); but this is false for characteristic 2. However, in 1953, San Soucie [10] showed that division rings of characteristic 2 which satisfy (11.1) are alternative. Thus Theorem 3 is true.

Instead of elaborating here the properties of Cayley division algebras, we refer the reader to an elementary discussion of these algebras from an entirely different point of view (Dickson [11]). It seems of more importance to indicate the geometric significance of Theorems 2, 3, 4. Hilbert [1] shows that if the Euclidean plane $\pi$ satisfies the vector axiom and that strong form of the distributive axiom obtained by omitting (*) (or, in standard language, if $\pi$ is Desarguesian) then, and only then, every planar ternary ring of $\pi$ is an associative division ring. Since the class of all division rings with the right inverse property turns out to be very little more extensive than the class of all associative division rings, we draw the following conclusion:

*If we intend to insist that every planar ternary ring of the Euclidean plane $\pi$ be a division ring, we may as well go the whole way and require $\pi$ to be Desarguesian.*

**12. Other points of view.** It would be unjust to leave the present topic without some brief reference to a great mass of literature entirely neglected here. Most of this literature (including Hall [2], Veblen and Wedderburn [4]) is written in the language of projective rather than Euclidean planes, which did not suit my aims.* But of course the study of Desarguesian planes, for example, did not originate with Hilbert.

* I shall make no serious attempt to link the present discussion to projective geometry. The reader can discover how to do this for himself by first reading §2 of Hall [2] through the first three lines of p. 232 and then considering Hall's Figure 4 (p. 264) together with the statement (below the figure) of the projective axiom Theorem L. First, in Hall's Figure 4, delete line $AMNB$ and its points and then carefully redraw the figure in the resulting Euclidean plane so that parallel lines

More particularly, the geometric meaning of alternative division rings was first studied by Ruth Moufang (see the references in [2] or [6]) and characterized by the uniqueness of the projective construction of a fourth harmonic point. Later Hall [2] gave an independent characterization in terms of his Theorem L. Theorem L is a projective axiom which, in two of its Euclidean forms, becomes* respectively the vector axiom and the distributive axiom. The characterization of Veblen-Wedderburn systems is originally due to Hall.

One final remark. It is an amusing fact that Theorem 6.4 of Hall [2], although true, was not completely proved until 1953, just ten years after the publication date of Hall's paper. By a trivial slip, the theorem contains the word "two" where "three" would have been appropriate. In 1950 I had a rude awakening in this connection which led to Theorem 2. The record shows that a similar experience led Skornyakov to the study of right alternative division rings.—May there be more such slips!

*Appendix I. Planar ternary rings.* A planar ternary ring is a system $(R, F)$ consisting of a set $R$ and a ternary operation $F$ subject to the following postulates:

(i) 0 and 1 are two distinct elements of $R$.

(ii) If $a, b, c$ are in $R$, $F(a, b, c)$ is a uniquely defined element of $R$.

(iii) $F(0, b, c) = F(a, 0, c) = c$ for all $a, b, c$ of $R$.

(iv) $F(a, 1, 0) = F(1, a, 0) = a$ for each $a$ in $R$.

(v) If $b, b', c, c'$ are in $R$, with $b \neq b'$, the equation $F(x, b, c) = F(x, b', c')$ has a unique solution $x$ in $R$.

(vi) If $a, a', b, b'$ are in $R$, with $a \neq a'$, the system of equations $F(a, x, y) = b$, $F(a', x, y) = b'$ has a unique solution $x, y$ in $R$.

(vii) If $a, b, c$ are in $R$, the equation $F(a, b, x) = c$ has a unique solution $x$ in $R$.

A planar ternary ring $(R, F)$ determines a unique Euclidean plane defined as follows: The points of the plane are the ordered pairs $(x, y)$ of elements $x, y$ of $R$. Each ordered pair $[m, b]$ of elements $m, b$ of $R$ is a line of the plane which passes through those points $(x, y)$ such that $y = F(x, m, b)$. Each symbol $[a]$, $a$ in $R$, is a line of the plane which passes through those points $(x, y)$ such that $x = a$.

*Addition* is defined for a planar ternary ring $(R, F)$ by $a + b = F(a, b, 0)$. The system $(R, +)$ is a loop. That is:

(viii) In the equation $x + y = z$, if any two of $x, y, z$ are assigned as elements of $R$, the third is uniquely determined as an element of $R$.

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(*e.g.*, *RS*, *XY*) appear parallel. You should recognize the figure for the vector axiom (triangles *ZXY*, *TRS*). Then note that Theorem L can be interpreted as the vector axiom: *ZY* is parallel to *TS*. Now begin afresh with Hall's Figure 4 (or, equivalently, restore the deleted line *AMNB* and its points). This time delete line *ARX* and its points and apply the same process. Theorem L, as now interpreted, gives a statement about triangles *NZT*, *MYS* which is slightly different from but clearly equivalent to the distributive axiom.—As an alternative suggestion, the reader may prefer to consult a pamphlet by H. G. Forder [12] which (I am told—I have not yet seen it) covers much the same ground as the present paper with more emphasis on the projective formulation.

* See footnote pp. 15–16.

(ix) There exists an element 0 of $R$ such that $0+a=a+0=a$ for every $a$ in $R$.

A loop $(R, +)$ is a group provided:

(x) $(a+b)+c=a+(b+c)$ for all $a, b, c$ of $R$, and is an abelian group if also

(xi) $a+b=b+a$ for all $a, b$ of $R$.

*Multiplication* is defined for a planar ternary ring $(R, F)$ by $ab=F(a, b, 0)$. In particular,

(xii) $0a=a0=0$ for all $a$ in $R$.

If $R^*$ consists of $R$ with 0 removed, $(R^*, \cdot)$ is a loop; that is, (viii), (ix) hold with $R, +, 0$ replaced by $R^*, \cdot, 1$ respectively.

*Appendix II. Special planar ternary rings.* A Veblen-Wedderburn system is a system $(R, +, \cdot)$ consisting of a set $R$ and two binary operations $+, \cdot$, subject to the following postulates:

(I) $(R, +)$ is an abelian group with zero 0.

(II.1) $(a+b)c=ac+bc$ for all $a, b, c$ of $R$.

(III) $(R^*, \cdot)$ is a loop with identity 1.

(IV) $a0=0$ for each $a$ of $R$.

(V) If $a, a', b$ are in $R$, with $a \neq a'$, the equation $xa=xa'+b$ has a unique solution $x$ in $R$.

A Veblen-Wedderburn system $(R, +, \cdot)$ becomes a planar ternary ring $(R, F)$ when $F$ is defined by $F(a, b, c)=ab+c$.

A division ring (with identity element 1) is a system $(R, +, \cdot)$ which satisfies (I), (II.1), (III) and

(II.2) $c(a+b)=ca+cb$ for all $a, b, c$ of $R$.

Every division ring (with identity) is a Veblen-Wedderburn system, but not conversely.

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# FINITE PROJECTIVE PLANES*

MARSHALL HALL, JR., The Ohio State University

**1. Coordinates.** We may introduce coordinates [6] in a projective plane in the following way: We take four points $X$, $Y$, $O$, $I$ no three of which are on a line.

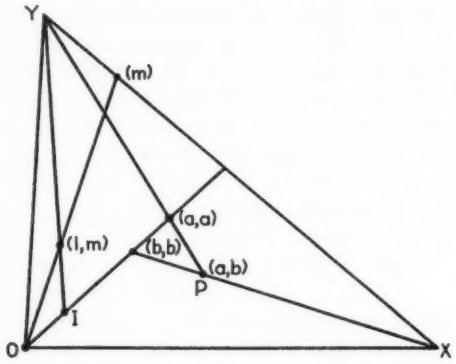


FIG. 1

We call $XY$ the line at infinity, $O=(0, 0)$ the origin, $I=(1, 1)$ the unit point, $OX$ the $x$-axis and $OY$ the $y$-axis. For the finite points on $OI$ besides (0, 0) and (1, 1) we assign further coordinates $(a, a)$, $(b, b)$ using a set $S$ of distinct symbols including 0 and 1. If $P$ is any finite point and $PY$ intersects $OI$ in $(a, a)$ and $PX$ intersects $OI$ in $(b, b)$ assign to $P$ the coordinates $(a, b)$. This rule reassigns the same coordinates to the points of $OI$. So far no coordinates have been assigned to points on the line at infinity. The line through (0, 0) and $(1, m)$ will cut the line at infinity in some point to which we assign the slope coordinate $(m)$. This assigns (0) to $X$, (1) to the intersection of $OI$ and $XY$ but leaves $Y$ unassigned. We may represent $Y$ by $(\infty)$.

A line through $Y$ contains those points $(x, y)$ with $x=c$, a constant. A line through $X$ will be $y=c$. The lines define operations on the elements of $S$ in a natural way. If $(x, y)$ is on the line joining $(0, b)$ and (1) we put $y=x+b$ to define addition. If $(x, y)$ is on the line joining (0, 0) and $(m)$ we put $y=xm$ to define multiplication. More generally, if $(x, y)$ is on the line joining $(0, b)$ and $(m)$, we put $y=x\cdot m \circ b$, defining a ternary operation on the elements of $S$. The ternary operation includes both addition and multiplication as special cases.

$S$ as a ternary ring satisfies the following laws which are consequences of the axioms for a projective plane:

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* Presented to the Mathematical Association of America at Pittsburgh, Pennsylvania, Dec. 30. 1954.

T1 $0 \cdot m \circ c = a \cdot 0 \circ c = c$, $1 \cdot m \circ 0 = m$, $a \cdot 1 \circ 0 = a$.

T2 $x \cdot m \circ b = c$ for given $m \neq 0$, $b$, $c$, has a unique solution $x$.

T3 $a \cdot m \circ z = c$ for given $a$, $m$, $c$ has a unique solution $z$.

T4 $a \cdot z \circ b = c$ for given $a \neq 0$, $b$, $c$, has a unique solution $z$.

T5 $x \cdot m_1 \circ b_1 = x \cdot m_2 \circ b_2$ has a unique solution $x$ if $m_1 \neq m_2$.

T6 The pair of equations $x_1 \cdot m \circ b = y_1$, $x_2 \cdot m \circ b = y_2$ has a unique solution for $m$ and $b$ if $x_1 \neq x_2$, $y_1 \neq y_2$.

These rules are dependent (in particular T2 follows from T1 and T5). Every choice of an ordered quadrilateral $X$, $Y$, $O$, $I$ in a plane leads to a ternary ring and conversely a ternary ring satisfying T1, $\cdots$, T6 determines a projective plane.

The different choices for the ordered quadrilateral will in general lead to non-isomorphic ternary rings. Any such isomorphism will determine a collineation of the plane and since there are planes with no collineations these planes have all their ternary rings different.

Any further properties of the ternary ring not given by T1 $\cdots$ T6 will correspond to configuration theorems. An important case is that in which the lines have linear equations: $y = x \cdot m \circ b = xm + b$. This relation corresponds to

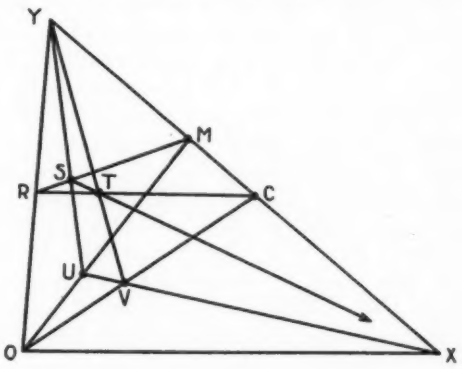


FIG. 2

the minor theorem of Desargues which I call Theorem L. Here $Y$, the center of perspectivity for the triangles $RST$ and $OUV$ lies on the axis of perspectivity $MCX$. As given here the line at infinity plays a special role and Theorem L is therefore called an affine theorem. In fact it is even more specialized in that $O$, $X$, $Y$ and $C$ all are fixed points.

If Theorem L is taken as a universal configuration theorem then the plane is coordinatized by an alternative division ring, in which multiplication satisfies the weak associative laws: $a(ab) = (aa)b$ and $b(aa) = (ba)a$. This may also be characterized by the weak associative laws: $a^{-1}(ab) = b = (ba)a^{-1}$. For a division ring the one-sided laws $a^{-1}(ab) = b$ and $a(ab) = (aa)b$ are equivalent if the char-

acteristic is not two; but if it is two, the law $a^{-1}(ab)=b$ is stronger. In fact it has been shown recently that a division ring with both distributive laws and the single law $a^{-1}(ab)=b$ is alternative. Geometrically this means that if Theorem L is satisfied for all choices of $XYMC$ on two different lines, then it is satisfied universally. This is a very deep result and has not been shown by any purely geometric means.

In his classical studies on the Foundations of Geometry, Hilbert [10] showed that coordinatization by an associative division ring was equivalent to the universal validity of the full theorem of Desargues and was implied by the affine form in which the axis of perspectivity is the line at infinity. Hilbert also showed that if in addition the Theorem of Pappus was valid then equivalently the division ring would be a field. It was shown by Hessenberg [11] that the Theorem of Pappus implies the Theorem of Desargues. Geometrically the Theorem of Pappus is equivalent to the assertion that in every ternary ring for a plane the multiplication is commutative, whence this result shows that all further field properties of the ternary rings follow from the commutativity of multiplication. Naturally the commutativity of multiplication in one ternary ring is a relation of a different kind in another.

The uniqueness of harmonic conjugates was shown by Monfang [17] to be equivalent, for characteristic not two, to coordinatization from an alternative division ring. Perhaps the most striking recent result in this subject is the discovery by Bruck and Kleinfeld [5] and independently by Skornyakov [25] that an alternative division ring is either (1) associative or (2) a Cayley-Dickson algebra over its center. Different degrees of specialization of Theorem L have been studied recently by Herbert Naumann [18] who has shown these to be equivalent to various properties of addition, multiplication, and distributivity. A ring with addition, a group, and one distributive law is called a *near-ring;* and if there are no divisors of zero, it is called a *near-field*. All finite (associative) near-fields have been determined by Zassenhaus [27]. A near-field is a special case of a *quasi-field* or Veblen-Wedderburn system. A quasi-field has abelian addition and one distributive law $(x+y)m=xm+ym$, but no further properties are assumed for multiplication other than that it be a loop. A system opposite to the quasi-field is the neo-field, investigated by Lowell Parge [20] and D. R. Hughes [13]. Here, multiplication is taken to be a group and the distributive laws are assumed, but no properties are required for the additive loop. Some infinite planar neo-fields are known, but no finite ones are known, except for finite fields.

An interesting unsettled question in the study of coordinates and configurations is the matter of the consequences of the Fano configuration. This says that the diagonal points of every quadrilateral are collinear. This is the configuration for characteristic two and is certainly satisfied in any alternative plane of characteristic two. But it is not known if the Fano planes are a still more general class. Rashevskii [22] has made some elementary investigations into this.

**2. Finite planes.** We may take $y=x'm \circ b=xm+b$ with coordinates from any field, and so, if we wish, from any finite field with $n=p^r$ elements. In such a projective plane every line contains $n+1$ points, and every point is on $n+1$ lines. In fact if for some finite $n \geqq 2$, one line of a plane contains $n+1$ points, it is easy to show that every line contains exactly $n+1$ lines, and the total number of points is $N=n^2+n+1$, and the total number of lines is $N=n^2+n+1$. Since every finite associative division ring or even finite alternative division ring is a finite field, in a finite plane Theorem L, the full Desargues theorem, and the theorem of Pappus are all equivalent if taken as universal theorems. Thus, for every finite Desarguesian plane with $n+1$ points on a line, $n$ is a prime power $n=p^r$. Although several different types of non-Desarguesian planes are known, in every case $n$ is a prime power. Professors Bruck and Ryser [4] have shown that a plane cannot exist if $n \equiv 1, 2 \pmod 4$, $n \neq a^2+b^2$. But the gap between this restriction and the prime powers is considerable. The first critical value of $n$ is $n=10$. A thorough investigation of this case is currently beyond the facilities of computing machines.

For $n=2, 3, 4, 5$ it is easy to test all cases and show that only the Desarguesian plane exists. The case $n=6$ is excluded by Bruck and Ryser. But this case had earlier been eliminated by Tarry [26] and stood for a long time as a solitary curiosity. This study was in terms of orthogonal latin squares. A *latin square* is an $n$ by $n$ square of $n^2$ cells filled with $n$ letters $a_1, a_2, \cdots, a_n$ so that each row contains each letter exactly once and also each column contains each letter exactly once. Two squares are said to be *orthogonal* if, when superposed, the ordered pairs $a_ib_j$ associated with the cells give each of the $n^2$ possibilities exactly once. A set of $n-1$ mutually orthogonal squares determines an affine plane (and so a projective plane) in this way. Regard the $n^2$ cells as points. Then the points are arranged in lines in the following ways: the $n$ points forming each row are on a line, the $n$ points forming each column are on a line, and each square determines $n$ lines, namely for each letter $a$ the $n$ points to which it assigns $a$. Conversely, an affine plane determines $n-1$ mutually orthogonal squares. Tarry showed that for $n=6$ there were not even two mutually orthogonal squares. Thus *a fortiori* there is no plane. It has been conjectured that for $n \equiv 2 \pmod 4$ there are no orthogonal squares. A proof of this based on topological considerations was attempted by MacNeish but the proof contains a major error. It has been shown by H. B. Mann [15] that for $n \not\equiv 2 \pmod 4$ there are at least two mutually orthogonal squares. More precisely for any $n=p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r}$ there are at least min $(p_i^{e_i}-1)$, $i=1 \cdots r$, mutually orthogonal squares. For $n=10$ extensive searches on the SWAC machine at UCLA have failed to produce any orthogonal squares. But even 100 machine hours will not cover more than a microscopic part of the complete search.

A pair of orthogonal latin squares is associated with a quadratic representation. We associate variables $x_i$ with rows, $y_j$ with columns, $z_k$ with letters of the first square and $w_s$ with letters of the second square. Then if $z_k$ and $w_s$ are in the cell of $x_i$, $y_j$, we form $(x_i+y_j+z_k+w_s)^2=Z$. Here

$$Q = n(\sum x_i^2 + \sum y_j^2 + \sum z_k^2 + \sum w_s^2) + 2\sum x_i \sum y_j + \cdots + 2\sum z_k \sum w_s$$

expresses precisely the conditions for the orthogonality. In this connection an interesting situation has arisen. Here $Q$ is the sum of squares of non-negative linear forms. As such it is surely semi-definite and has non-negative coefficients. I raised the question as to whether in general there were further conditions on a sum of squares of non-negative forms. Professor Alfred Horn showed that further conditions do exist and that in the theory of convex spaces these forms are adjoint to forms which are non-negative for non-negative arguments. On 5 or more variables further conditions exist.

For $n=7$ the only plane is the Desarguesian plane. There are 147 essentially different $7\times 7$ latin squares of which 146 were listed by Norton [19] and the missing 147th was found by Sade [22]. Finding 6 mutually orthogonal squares gives only the Desarguesian plane. A more direct counting method by Pierce [21] and M. Hall [8] also establishes this result. Calculations on the SWAC machine show that the Desarguesian plane is the only one for $n=8$.

For $n=9$ there are several non-Desarguesian planes. Albert [1] has shown the existence of non-Desarguesian planes for $n=p^r$, $p$ odd, $r\geqq 2$. There are also non-Desarguesian planes for $n=2^{2^s}$, $s\geqq 2$. For $n=p$ only the Desarguesian plane is known. Near-fields and Veblen-Wedderburn systems have given many non-Desarguesian planes.

If addition is an elementary abelian group, then $n=p^r$. Thus to find a plane with $n\neq p^r$ we must not restrict the addition too heavily. In a neo-field multiplication is taken as an abelian group and the distributive laws are assumed, but no further direct property of addition. A number of additional properties have been proved for finite neo-fields. Unfortunately no planar neo-field has been found yielding anything but Desarguesian planes in the finite case.

A further attempt to construct finite planes is in terms of a collineation group. Let us assume that a plane has a collineation of order $N=n^2+n+1$, cyclic on the points. It will also be cyclic on the lines of the plane and the points of a line will yield a difference set. Thus 0, 1, 3, 9 are a difference set modulo 13 since every $d\not\equiv 0 \pmod{13}$ is of the form $a_i-a_j\equiv d \pmod{13}$ exactly once with the $a$'s from the set 0, 1, 3, 9. Here $i+0$, $i+1$, $i+3$, $i+9 \pmod{13}$, $i=0, \cdots, 12$ give the lines of the plane with $n=3$. More generally with $N=n^2+n+1$, residues $a_0, a_1, \cdots, a_n \pmod N$ form a difference set if $d\not\equiv 0 \pmod N$ is expressible exactly once in the form $d\equiv a_i-a_j \pmod N$. The construction of difference sets is precisely equivalent to constructing cyclic planes. Singer [24] showed that every finite Desarguesian plane is cyclic and unfortunately no others have been found. Evans and Mann [16] have shown that for a cyclic plane we must have $n=p^r$ at least for $n\leqq 1600$. The cyclic approach has at least yielded some results of interest for symmetric designs. If $k(k-1)=d(v-1)$ and $a_1, a_2, \cdots, a_k \pmod v$ are such that for $d\equiv 0 \pmod v$ the congruence $a_i-a_j\equiv d \pmod v$ has exactly $\lambda$ solutions, then the sets $a_1+j, \cdots,$

$a_k+j \pmod v$, $j=0, \cdots v-1$, will form the blocks of a symmetric design.

In constructing the cyclic planes and cyclic designs, a *multiplier* is very useful. The number $t$ is a multiplier if $ta_1, ta_2, \cdots, ta_k \pmod v$ are $a_1+s, \cdots, a_k+s \pmod v$ in some order. It is easy to see that the multipliers are prime to $v$ and form a multiplicative group modulo $v$. I have proved that if $p \mid k-\lambda=n$, $(p, v)=1$ and $p>\lambda$, then $p$ is a multiplier. For the planes, since $\lambda=1$, the condition $p>\lambda$ is trivial. Actually $p>\lambda$ seems by examples to be an unnecessary condition and also out of place in a purely arithmetical situation, but so far the proofs of the existence of multipliers require this. Here for $K=9$, $v=37$, $\lambda=2$, we have $n=7$ and the solution is 1, 7, 9, 10, 12, 16, 26, 33, 34 (mod 37). Although there have been found only the Desarguesian planes among the finite cyclic planes it is conceivable that there may be others. The cyclic designs are not necessarily unique. There are two cyclic designs for $K=15$, $v=31$, $\lambda=7$, and two for $K=21$, $v=43$, $\lambda=10$.

R. H. Bruck* has generalized to replace the cyclic group by any abelian group and A. Hoffman [12] has considered cyclic affine planes, these being planes with a collineation fixing $L$ and $O$ and cyclic on the remaining $n^2-1$ points. In both cases the multiplier theorem carries over.

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# GEOMETRIES AND INCIDENCE MATRICES*

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**1. Introduction.** A projective plane $\pi$ is a mathematical system composed of elements, called "points," and certain sets of points, called "lines," subject to the following postulates:

P (1) *Two distinct points are contained in one and only one line.*

P (2) *Two distinct lines contain one and only one point in common.*

P (3) *There exists at least one set of four points of $\pi$, no three of which are contained in one line.*

Other equivalent sets of postulates for a projective plane are available. Those which we have selected appear in a now classical paper by M. Hall [6]. Briefly, postulates P (1) and P (2) are entirely basic to the system. Postulate P (3) serves to exclude certain degenerate systems, which satisfy only P (1) and P (2). The recent expositions of Skornyakov [21] and Hall [7] lend considerable insight into the geometrical consequences of these postulates. A book by Günter Pickert concerning projective planes is scheduled to appear soon.

Let us now impose a further restriction on $\pi$—one that at first glance appears to be quite exotic. Suppose that the total number of points of $\pi$ is finite. Such planes are called *finite planes* and these systems play an important role in modern combinatorial analysis. If $\pi$ is finite, then one proves without great difficulty that there exists a positive integer $n \geqq 2$ with the following properties:

(1) *Each line of $\pi$ contains exactly $n+1$ distinct points.*

(2) *Each point of $\pi$ is contained in exactly $n+1$ distinct lines.*

(3) *The plane $\pi$ is made up of exactly $n^2+n+1$ distinct points and $n^2+n+1$ distinct lines.*

The integer $n$ is clearly an important invariant of the finite plane. The first and most primitive problem that arises is the determination of the precise range of values of $n$ for which finite planes exist. This problem is largely unsolved at the present time. Consider a finite Galois field of $n=p^\beta$ elements, where $p$ is a prime and where $\beta$ is a positive integer. By utilizing the properties of these finite fields, one readily constructs a finite plane $\pi$, with $n+1$ points per line, in which the Theorem of Desargues is valid. Finite non-Desarguesian planes have also been constructed by various methods, but in all known cases, we have $n=p^\beta$.

Thus it is natural to conjecture that planes $\pi$ do not exist for values of $n$ other than $n=p^\beta$. The investigations of Tarry imply the correctness of the conjecture for the case $n=6$ [22]. Concerning this conjecture, one may establish the following Theorem of Bruck and Ryser on the nonexistence of finite planes [2].

* Presented to the Mathematical Association of America at Pittsburgh, Pennsylvania, on December 30, 1954.

THEOREM 1. *Let $n \equiv 1$ or 2 (mod 4) and let the squarefree part of $n$ contain at least one prime factor of the form $4k+3$. Then $\pi$ with $n+1$ points on a line does not exist.*

Evidently Theorem 1 excludes geometries for infinitely many values of $n$, for example, the values $n=2p$, where $p$ is a prime of the form $4k+3$. Of course, there remain infinitely many undecided values. No values of $n$ other than those covered by Theorem 1 have been excluded up to the present time. This state of affairs has resulted in the development of two opposing points of view. Conservatives maintain $\pi$ exists if and only if $n=p^{\beta}$, and liberals maintain $\pi$ exists for all values of $n$, except those specifically excluded by Theorem 1. The smallest undecided value is $n=10$.

**2. The $v$, $k$, $\lambda$ problem.** Let it be required to arrange $v$ elements into $v$ sets such that every set contains exactly $k$ distinct elements and such that every pair of sets has exactly $\lambda$ elements in common, $0<\lambda<k<v$. This problem we refer to as the $v$, $k$, $\lambda$ *problem*, and we call the resulting arrangement a $v$, $k$, $\lambda$ *configuration* [3]. These configurations are actually a generalization of the finite planes $\pi$ described in the introduction.

Let us first consider an example of a $v$, $k$, $\lambda$ configuration. Denote the elements by $x$ and the sets by $T$, and let

$$T_1 = \{x_1, x_2, x_4\},$$

$$T_2 = \{x_2, x_3, x_5\}, \quad T_3 = \{x_3, x_4, x_6\}, \quad T_4 = \{x_4, x_5, x_7\},$$

$$T_5 = \{x_5, x_6, x_1\}, \quad T_6 = \{x_6, x_7, x_2\}, \quad T_7 = \{x_7, x_1, x_3\}.$$

It is easy to verify that this is a $v$, $k$, $\lambda$ configuration for the case $v=7$, $k=3$, and $\lambda=1$.

For a $v$, $k$, $\lambda$ configuration, let us list the elements $x_1, \cdots, x_v$ in a row, and the sets $T_1, \cdots, T_v$ in a column. Insert 1 in row $i$ and column $j$ if $x_j$ belongs to set $T_i$, and 0 otherwise. In this way we obtain a $v$ by $v$ matrix $A$ of 0's and 1's, called the *incidence matrix* of the $v$, $k$, $\lambda$ configuration. This matrix contains all the essential information given by the $v$, $k$, $\lambda$ configuration.

THEOREM 2. *The $v$, $k$, $\lambda$ problem has a solution if and only if there exists a 0, 1 matrix $A$ of order $v$ such that*

$$AA^T = B,$$

*where $A^T$ denotes the transpose of $A$ and where the matrix $B$ has $k$ in the main diagonal and $\lambda$ in all other positions.*

Theorem 2 follows directly from the definition of a $v$, $k$, $\lambda$ configuration.

THEOREM 3. *For a $v$, $k$, $\lambda$ configuration,*

$$\lambda = \frac{k(k-1)}{v-1}.$$

Theorem 3 appears in [16]. A generalization may be found in [10]. To prove the result, let $s_i$ denote the sum of column $i$ of the matrix $A$. Then for $i=1, 2, \cdots, v$,

$$a_{i1}s_1 + \cdots + a_{iv}s_v = k + (v-1)\lambda. \tag{4}$$

The determinant of this system is $|A|$. Since $AA^T=B$, and

$$|B| = (k-\lambda)^{v-1}(k+(v-1)\lambda),$$

we have $|A| \neq 0$. Thus the system (4) has a unique solution, $s_i=c$. But $s_1+\cdots+s_v=kv$, and hence $c=k$. Thus $k^2=k+(v-1)\lambda$.

THEOREM 4. *The incidence matrix of a $v$, $k$, $\lambda$ configuration is normal, that is*

$$AA^T = A^TA.$$

For we may let

$$Q = \begin{bmatrix} -k & \sqrt{-\lambda} & \cdots & \sqrt{-\lambda} \\ \sqrt{-\lambda} & & & \\ \vdots & & A & \\ \sqrt{-\lambda} & & & \end{bmatrix}.$$

Since $\lambda=k(k-1)/(v-1)$, we have $QQ^T=(k-\lambda)I$, where $I$ is the identity matrix of order $v+1$. But then $Q/\sqrt{k-\lambda}$ is orthogonal, and hence $QQ^T=Q^TQ$. But because of the structure of $Q$, $AA^T=A^TA$.

In a $v$, $k$, $\lambda$ configuration, let $\lambda=1$, and let $k=n+1$, where $n\geqq 2$. Then $v=n^2+n+1$, and the $v$, $k$, $\lambda$ configuration reduces to a projective plane $\pi$ with $n+1$ points per line. Indeed, the normality of the incidence matrix $A$ assures us that the principle of duality operates in the finite plane.

Other specializations of the values of $v$, $k$, and $\lambda$ lead to other classical types of configurations. For example, for $v=4m-1$, $k=2m-1$, $\lambda=m-1$, the $v$, $k$, $\lambda$ configuration is equivalent to a Hadamard matrix of order $N=4m$ [15]. These are the $\pm 1$ matrices $H$ satisfying $HH^T=NI$. Here $H$ is of order $N$ and $I$ denotes the identity matrix. If $H$ exists, then it is easy to show that the order of $H$ is 1, 2, or $\equiv 0 \pmod 4$. It is conjectured that $H$ exists for each of these orders. The first undecided value is $N=92$. The $v$, $k$, $\lambda$ configurations arise in a natural way also in statistics, where they are referred to as symmetrical block designs. An extensive literature has developed concerning these designs and their generalizations (see, for example [12, 11]).

Finally, it should be mentioned that we have imposed no restrictions on the incidence matrix $A$. If, for example, we require the incidence matrix to be cyclic, then the $v$, $k$, $\lambda$ configuration reduces to the perfect difference set which arises in the theory of numbers [20, 8]. For geometries, this restriction is roughly equivalent to the validity of the Theorem of Desargues [5].

**3. Nonexistence theorems.** In this section we summarize some of the main results obtained by investigating the matric equation $AA^T = B$, where $B$ has $k$ in the main diagonal and $\lambda$ elsewhere.

THEOREM 5. *If the $v$, $k$, $\lambda$ problem has a solution and if $v$ is even, then $k-\lambda$ must be a square.*

This result follows easily upon taking determinants in the matric equation $AA^T = B$ [18, 3].

THEOREM 6. *If the $v$, $k$, $\lambda$ problem has a solution and if $v$ is odd, then*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

*possesses a nonzero integral solution.*

Theorem 6 appears in [3] and is derived by methods which are entirely elementary, requiring nothing more than the fact that every positive integer is the sum of four rational squares. For further commentaries concerning [3], see [7, 8, 11]. If $\lambda = 1$, then Theorem 6 reduces to Theorem 1. Indeed, Theorem 6 may be derived along lines entirely analogous to the original proof of Theorem 1 appearing in [2], which utilized the Minkowski-Hasse invariants of a quadratic form. In this connection see [19].

Theorems 5 and 6 for $v$, $k$, $\lambda$ configurations give us no new information concerning geometries, and they tell us nothing about the state of affairs for Hadamard matrices. They do, however, effectively exclude numerous $v$, $k$, $\lambda$ configurations. Indeed, up to now these are the only $v$, $k$, $\lambda$ configurations excluded, so that one may conjecture their existence in all other cases.

One may show that the values of $v$, $k$, $\lambda$ excluded by Theorems 5 and 6 are precisely those for which there is no rational $X$ such that $XX^T = B$. Thus the nonexistence theorems have not utilized the normality of the incidence matrix, and it is natural to inquire if this additional restriction on $A$ cannot be used to our advantage. Concerning this point, one may prove the following generalization of a Theorem of Albert [1, 9].

THEOREM 7. *Suppose there is a rational $X$ such that $XX^T = B$. Then there is a rational $A$ such that $AA^T = A^TA = B$.*

Finally, Theorem 7 may be extended to yield the following [9]:

THEOREM 8. *Suppose there is a rational $X$ such that $XX^T = B$. Let $A_1$ be an $r$ by $v$ matrix of 0's and 1's, such that*

$$A_1A_1^T = B_1,$$

*where $B_1$ is of order $r$, and has $k$ in the main diagonal and $\lambda$ in all other positions. Then there is a $v$ by $v$ rational $A$ having $A_1$ as its first $r$ rows such that*

$$AA^T = A^TA = B.$$

Thus any consistent 0, 1 "start" for an incidence matrix may always be completed both rationally and normally. These results are in sharp contrast to those of Connor, who has investigated non-symmetrical designs, and obtained restrictions on initial blocks [4].

**4. An integral approach.** It seems reasonably certain that arguments restricted to the rational field cannot by themselves answer all the questions arising in the study of the existence of these configurations. We turn then to a study of the equation $XX^T=B$, where we suppose that $X$ has *integral* elements. The results obtained up to now concerning integral investigations are a good deal more sketchy. One may easily derive the following [17].

THEOREM 9. *Suppose that $AA^T=A^TA=B$, where $A$ is integral. Then $A$ is a 0, 1 matrix, or a 0, $-1$ matrix, and hence yields a $v$, $k$, $\lambda$ configuration.*

The requirement of the normality of $A$ in Theorem 9 is annoying, and may be removed for some values of $v$, $k$, $\lambda$ [17].

THEOREM 10. *Let $(k, k-\lambda)=1$ and let $k-\lambda$ be odd. Suppose that $AA^T=B$, where $A$ is integral. Upon multiplication of the columns of $A$ by $\pm 1$, $A$ is transformed into a 0, 1 incidence matrix.*

Thus, at least for the case of $(k, k-\lambda)=1$ and $k-\lambda$ odd, the $v$, $k$, $\lambda$ problem is equivalent to the problem of representing $B$ integrally by means of the identity. Unfortunately, this problem in the theory of quadratic forms is an unsolved one.

Theorem 10 is not valid for the case in which $k-\lambda$ is even. Consider the projective plane, with $n$ an even integer. For this situation, it appears that there will exist integral solutions of $XX^T=B$, provided only that $B$ is rationally congruent to the identity. To substantiate this conjecture, integral solutions have been exhibited for $n$ equal to the order of a Hadamard matrix. Also, an interesting integral solution has been obtained for the case $n=10$ [9]. These integral solutions, however, are far removed from 0, 1 incidence matrices.

**5. Future research.** The incidence matrix has shown itself to be a powerful tool in the study of geometries. Arithmetical properties of the matrix $A$ yield new insight into the geometry of $\pi$. These arithmetical properties may have little or no geometrical significance, and this constitutes an inherent weakness of the approach. However, it may also serve to explain in part why no purely geometrical derivation of Theorem 1 has as yet been obtained.

The problems remaining are both difficult and numerous. The most immediate center around an extension of Theorem 1. Other vital problems in geometry, however, also have an incidence matrix interpretation. For example, it has been conjectured that every finite plane possesses a nontrivial collineation. Relative to incidence matrices, this means that we are to find a permutation matrix $P\neq I$ and a permutation matrix $Q$ such that $PAQ=A$. Suppose that the finite plane $\pi$ has additional requirements imposed upon it of one kind or another.

These requirements must be reflected in the structure of the matrix $A$. In this way we encounter a large class of problems dealing with configuration theorems and the like. Doubtless some of these can be effectively handled by matric methods. Finally, one may attempt an enumeration of the distinct geometries for specified values of $n$. This problem, however, appears to be entirely beyond the range of our present machinery, except for small values of $n$. Each of the problems in question may be generalized to $v$, $k$, $\lambda$ configurations.

These problems are subject to a variety of attacks. If possible, the integral solutions of $XX^T=B$ should be more fully explored. Equally interesting would be a continuation of Pall's study of the elementary divisors of $A$ [14]. Concerning elementary divisors, we remark that the inverse of $A$ is equal to $[1/(k-\lambda)][A^T-(\lambda/k)S]$, where $S$ is the matrix of all 1's. This relationship allows one to compute the elementary divisors of $A$, under the restriction that $(k, k-\lambda)=1$ and that $k-\lambda$ is a squarefree integer.

An arbitrary incidence matrix $A$ may be written in the form $A=L_1+\cdots+L_k$, where the $L_i$ are permutation matrices. This is a consequence of the theory of systems of distinct representatives [13]. The relationship has not yet been utilized in the study of $v$, $k$, $\lambda$ configurations. The eigenvalues of $A$ are certainly not invariant under permutations of the rows and columns of $A$. Nevertheless, they may have a deep combinatorial significance. One eigenvalue of $A$ is $k$, and the remaining $v-1$ eigenvalues are algebraic integers, each equal to $\sqrt{k-\lambda}$ in absolute value. That the eigenvalues remain invariant in absolute value under arbitrary permutations of the rows and columns is in itself a remarkable property of $A$. Note also that in the cyclic case the eigenvalues equal the algebraic integers $\theta(\epsilon^i)$, and these integers are basic to the study of difference sets [8].

Yet another approach might center around the matric equation $XX^T=B$, where the elements of $X$ are nonnegative reals. Finally, one must mention modern computational procedures. For $n=10$, there are something like $2^{10,000}$ possible candidates for an incidence matrix. Machines cannot cope with numbers of this magnitude. But the right combination of computation and theory will continue to produce worthwhile results.

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# FINITE PLANE PROJECTIVE GEOMETRIES*

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**1. Introduction.** Only finite plane projective geometries will be discussed in this paper, and, for the sake of brevity, they will be called "geometries." A geometry with $n$ points on a line is easily constructed provided $n-1$ is a power of a prime. It is not known if other geometries exist. There is only one geometry with $n$ points on a line if $n=3$, 4, 5, 6, or $S$, but there are at least four geometries for $n=10$, one desarguesian and the other three non-desarguesian [2, p. 273, 3, p. 912, 1, p. 411]. (There may be more.) There are no geometries with seven points on a line. It is not known if there exists more than one geometry for the case $n=9$.

Some elementary properties of geometries follow easily from the postulates, and others are obtained when the geometry is constructed by means of an algebraic system. Any geometry may be constructed from a ternary system satisfying certain conditions and it is therefore of interest to examine the algebra of such a system.

**2. Definitions and elementary properties of a finite plane projective geometry** [1, p. 323]. A class of elements or *points* in which there is determined a class of subsets called *lines* satisfying the following conditions or axioms A is called a *finite plane projective geometry.*

A1. *S contains only a finite number of points.*

A2. *There exist a point and a line not incident.*

A3. *Each line is incident with at least three points.*

A4. *Any two distinct points are incident with one and only one line.*

A5. *Any two lines are incident with at least one point.*

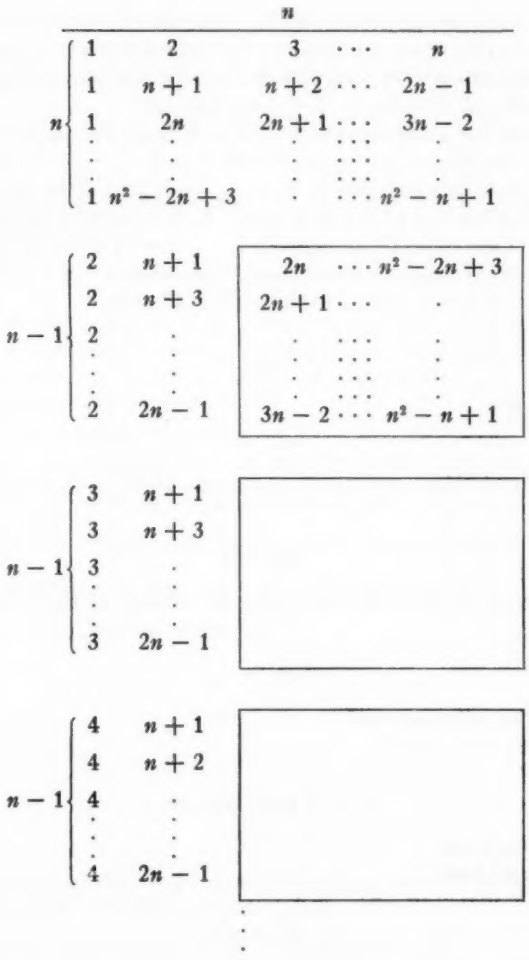
The above set of postulates is equivalent to the dual set; therefore, the principle of duality holds. Also, for a geometry with $n$ points on a line there are $n$ lines containing any point, and there are $n^2-n+1$ points (or lines) in the geometry.

**3. Rules for the construction of any geometry** [4, p. 9]. Let the first $n^2-n+1$ positive integers be the points of a geometry with $n$ points on a line. Let each line of the geometry be named by the $n$ integers which are points incident with the line.

Below is a skeleton which may be used as a starting point in attempting to construct a geometry.

Each row will represent a line, the elements of the row representing points incident with the line. There will be $n$ lines containing each integer.

* The author wishes to thank Professor E. Baylis Shanks of Vanderbilt University for his direction and encouragement in the writing of this paper.




Notice that the arrays inside the rectangles involve only the $(n-1)(n-2)$ consecutive integers $2n, \cdots, n^2-n+1$. Let these $n-1$ arrays, each consisting of $n-1$ rows and $n-2$ columns, be filled in with the same integers $2n$, $2n+1, \cdots, n^2-n+1$, such that the following rules are fulfilled:

I. *The j-th column of each array contains the same integers as the j-th column of the first array.*

II. *Each integer is in a different row in each array.*

III. *There is one row containing any two integers from different columns.*

Rule I is merely an agreement which does not affect the construction of the geometry materially, since the order of the points of a line is immaterial.

It is not difficult to show that the above collection of lines and points form a geometry if and only if rules I, II, III are obeyed.

If we include the column to the left of each array we have $n-1$ arrays, each $n-1$ by $n-1$. Let the arrays be numbered $0, 1, 2, \cdots, n-2$, and let the rows (columns) of each be numbered $0, 1, \cdots, n-2$. Let $a_{ij}^{k}$ be the element in the $i$-th row and $j$-th column of the $k$-th array. Rule I permits the elements in each column to be composed of the integers $0, 1, \cdots, n-2$ without loss of generality. Also the 0-th array may be considered to consist of $n-1$ identical columns, each containing $0, 1, \cdots, n-2$ in that order. That is,

$$a_{xj}^{0} = x. \tag{1}$$

We have also that

$$a_{x0}^{k} = x. \tag{2}$$

The element in the 0-th row and 1-st column of the $x$-th array may be taken as $x$. That is,

$$a_{01}^{x} = x. \tag{3}$$

And, the element in the 0-th row and $x$-th column of the 1-st array may be taken as $x$, or

$$a_{0x}^{1} = x. \tag{4}$$

Rule I gives the following rule:

(5) If $b \in \{0, 1, \cdots, n-2\}$,

$$a_{xj}^{k} = b \text{ has a solution } x.$$

From rule II we have:

(6) For $j \neq 0$ and $k \neq k'$,

$$a_{ij}^{k} \neq a_{ij}^{k'}.$$

Rule III gives the following:

(7) For $j \neq j'$, there exists a pair $x, y$ such that

$$a_{yj}^{x} = a_{ij}^{k} \quad \text{and} \quad a_{yj'}^{x} = a_{i'j'}^{k'}.$$

Now let $a_{ij}^{k}$ be considered as a ternary operation $jki$ on the elements of the set $0, 1, \cdots, n-2$, such that $jki$ is in the set. Equations (1) through (7) are then written

(8) $0ax = a0x = x$, $x10 = 1x0 = x$,

(9) $abx=c$ has a unique solution $x$,
(10) $axb=c$ has a unique solution $x$ if $a\neq 0$,
(11) For $a\neq c$, there exists a unique solution $x$, $y$ to the equations

$$\iota xy = b,$$
$$cxy = d.$$

Equations (8) and (9) follow from (1), (2), (3), (4), (5). Equation (6) is equivalent to the following:

$$\text{For } a \neq 0 \quad \text{and} \quad c \neq d, \qquad adb \neq acb.$$

This condition is equivalent to (10). Equation (11) follows from (7).

One of the above postulates is redundant, and consequently this *H-system* may be defined as one with a finite number of elements $0, 1, a, b, \cdots$ such that:

H1. $a0b=0ab=b$, $1a0=a10=a$.

H2. *$abx=c$ has a unique solution $x$.*

H3. *For $a\neq c$, the equations $axy=b$, $cxy=d$ have a unique solution $x$, $y$.*

THEOREM 3.1. *If $c\neq 0$, then equation $cxb=d$ has a unique solution $x$.*

*Proof.* In H3 if $a=0$ and $c\neq 0$, the postulate gives the desired result.

LEMMA. *If $a\neq c$, $x_1\neq x_2$, and $x_1ab=x_1cd$, then $x_2ab\neq x_2cd$.*

*Proof.* Suppose the contrary and let

$$x_1ab = x_1cd = f,$$
$$x_2ab = x_2cd = g.$$

Then the equations

$$x_1xy = f,$$
$$x_2xy = g,$$

have two solutions, namely $x=a$, $y=b$, and $x=c$, $y=d$. This violates H3.

Let the following postulate be called H3′.

H3′. *For $a\neq c$, $xab=xcd$ has a unique solution $x$.*

THEOREM 3.2. *The postulates* H1, H2, H3 *are equivalent to the set* H1, H2, H3′.

*Proof.* First it is shown that H1, H2, H3 imply H3′. If $a\neq c$, and $b=d$, the only solution of $xab=xcd$ is $x=0$. If $a\neq c$ and $b\neq d$, let $d_i$ be such that $x_iab=x_icd_i$, $(i=1, 2, 3, \cdots)$, where $x_1, x_2, x_3, \cdots$ are distinct. It follows from the lemma that $d_1, d_2, \cdots$ are distinct. Hence, for any $d$ the equation $xab=xcd$ has a unique solution.

Next it is proved that H1, H2, H3′ imply H3. Consider the equations

$axy=b$, $cxy=d$, where $a\neq c$. For any $x_i$ there exists a $y_i$ such that $ax_iy_i=b$. (See H2). This gives a correspondence $x_i\rightarrow y_i$. For $x_i\neq x_j$, $cx_iy_i\neq cx_jy_j$, since the equation $xx_iy_i=xx_jy_j$ has but one solution $x=a$. (See H3′). Hence $cx_iy_i$ takes on the value $b$ for some unique $i$.

THEOREM 3.3. *In an H-system the equation $xab=c$ has a unique solution $x$ if $a\neq 0$.*

*Proof.* In H3′ let $0=c\neq a$. Then the theorem follows.

We sum up these results by the following theorem.

THEOREM 3.4. *Any H-system may be characterized by* H1, H2, H3, *or* H1, H2, H3′. *From each such system a geometry may be constructed. Conversely, any geometry leads to an H-system.*

Marshall Hall develops the postulates for a ternary system which characterizes projective geometries (finite or infinite) [2, p. 247]. In the finite case, his system is equivalent to the H-system, but the methods of construction of the ternary system from the postulates of the geometry differ in point of view and procedure.

It is of great interest to examine these ternary H-systems.

**4. Analysis of Hall's ternary system.** In this section are derived some theorems that are fundamental in a treatment of the H-system. All of the properties and theorems discussed correspond closely to some of the elementary properties and theorems related to binary systems. Some of the questions of interest are these:

What additional conditions must be fulfilled in order that the ternary system be abstractly equivalent to a double binary system? Does there exist an H-system with $r$ elements, where $r$ is not a prime-power? What are the characteristics of an H-system which is "distributive" or "associative," or both?

In an H-system the ternary operation $abc$ should not be assumed to be the result of two binary operations. But the following properties are easily shown to be true:

(1) $\quad abc = 0 \rightleftharpoons$ either $a = 0$, or $b = 0$.

(2) $\quad abd = acd,\ a \neq 0 \rightarrow b = c.$

(3) $\quad bad = cad,\ a \neq 0 \rightarrow b = c.$

(4) $\quad abc = abd \rightarrow c = d.$

Define for each element $x$ in an H-system

$$x_0 = 0 \quad \text{and} \quad x_i = 1x_{i-1}x$$

for $i=1, 2, \cdots$, and let the *A-order* of $x$ be the first $i\neq 0$ for which $x_i=0$. It is clear that if $x\neq 0$, $x_i\neq x_{i-1}$. Also, if $x_k=x_j$ ($k$ and $j$ positive) it follows that $x_{k-1}=x_{j-1}$. For $1x_{k-1}x=1x_{j-1}x$ implies $x_{k-1}=x_{j-1}$. (See property 2.)

THEOREM 4.1. *Each element of H has an A-order.*

*Proof.* The $A$-order of 0 is equal to 1. For $x \neq 0$, let $x_k$ be the first $x_i$ which is a repetition of some preceding one, say $x_j$. Then $x_0, x_1, x_2 \cdots, x_{k-1}$ are distinct and $x_k \neq x_1, x_2, \cdots, x_{k-1}$, or else $x_{k-1}$ is equal to one of the set $x_0, x_1, \cdots, x_{k-2}$. Hence $x_k = 0 = x_0$.

It is interesting to note that an $M$-order can be defined in such a way as to be analogous to the multiplicative order of an element in a field, just as the $A$-order already defined is analogous to the additive order in a field. We have merely to define $x^0 = 1$, $x^i = x^{i-1}x0$, for any $x \neq 0$ and every $x \neq 0$ will have an $M$-order.

THEOREM 4.2. *For an H-system any one of the following conditions is equivalent to each of the other two.*

(1) $$1x_ix_j = x_{i+j},$$

(2) $$1(1x_ix_j)x_k = 1x_i(1x_jx_k),$$

(3) $$1(1x_ix_j)x = 1x_i(1x_jx).$$

(Unless otherwise specified, operations among subscripts are modulo $m$, the $A$-order of the element.)

*Proof.* First it is shown that condition (1) implies condition (2). If $1x_ix_j = x_{i+j}$, then

$$1(1x_ix_j)x_k = 1(x_{i+j})x_k = x_{i+j+k} = 1x_ix_{j+k} = 1x_i(1x_jx_k).$$

It is obvious that condition (2) implies condition (3). For if $k$ is set equal to 1 in condition (2), condition (3) is obtained.

Next it is proved that condition (3) implies condition (1). An induction on $j$ is used. For $j = 1$, $1x_ix_j = x_{i+j}$ by definition. If $k$ is such that $1x_ix_k = x_{i+k}$, then

$$1x_ix_{k+1} = 1x_i(1x_kx) = 1(1x_ix_k)x = 1x_{i+k}x = x_{i+k+1},$$

and the induction is complete.

Since (1) implies (2), (2) implies (3), and (3) implies (1), the theorem is true.

THEOREM 4.3. *If one of the conditions in Theorem 4.2 is satisfied, then it follows that* $(x_i)_j = x_{ij}$.

*Proof.* An induction on $j$ is used. Notice that $(x_i)_1 = x_i = x_{i(1)}$, and let $k$ be such that $(x_i)_k = x_{ik}$. Then

$$(x_i)_{k+1} = 1(x_i)_kx_i = 1x_{ik}x_i = x_{ik+i} = x_{i(k+1)}.$$

THEOREM 4.4. *If either of the "distributive" laws*

(a) $ax_i0 = (ax0)_i$

(b) $x_ia0 = (xa0)_i$

*holds, then all non-zero elements of H have the same A-order.* (The hypothesis could be weakened by letting $x=1$ in (a) and (b)).

*Proof.* Suppose the contrary, that is, suppose there are elements $x$ and $y$ with $A$-orders $j$ and $k$ respectively, where $0<j<k$. Then if (a) holds, there exists an element $a\neq 0$ such that $ay0=x$ and

$$0 = x_j = (ay0)_j = ay_j0.$$

On the other hand, if (b) holds, there exists an element $a\neq 0$ such that $ya0=x$ and

$$0 = x_j = (ya0)_j = y_ja0.$$

In either case a postulate has been violated, for $ab0=0$ implies that either $a$ or $b$ must be 0.

The element $x$ in $H$ is said to generate the *m-subset* $X=\{x_1, x_2, \cdots\}$. The element $y$ generates the $m$-subset $Y$, *etc.*

THEOREM 4.5. *Let $H_1$ be any H-system such that one of the conditions of Theorem* 4.2 *and one of the "distributive" laws of Theorem* 4.4 *are satisfied. Then*

(a) *Any two m-subsets of $H_1$ are either identical or without common elements other than* 0;

(b) *the number N of elements in $H_1$ is*

$$N = n(p-1)+1,$$

*where n is the number of m-subsets and p is a prime equal to the A-order of each non-zero element.*

*Proof.* Since $(x_i)_j=x_{ij}$, any non-zero element of $X$ generates $X$, and any non-zero element of $Y$ generates $Y$. For (by Theorem 4.4) all non-zero elements have equal $A$-orders. If $x_i=y_j\neq 0$, then as $k$ varies $(x_i)_k=(y_j)_k$ generates $X$ and also $Y$. Hence set $X$ is the same as set $Y$. This completes the proof of (a).

Since $x_i\neq 0$ is a generator of the $m$-subset $X$ and $(x_i)_j=x_{ij}$, no product $ij$ (modulo $m$, the order of $x$) gives 0 unless either $i=0$ or $j=0$. Otherwise the $A$-order of $x_i$ would not be the same as $m$, the $A$-order of $x$. Hence $m$ is a prime $p$. It is seen that each of the $n$ $m$-subsets contains the element 0 and $p-1$ other elements. Since 0 is the only element common to two distinct $m$-subsets, $N=n(p-1)+1$.

THEOREM 4.6. *Let an H-system be given such that $(x_i)_j=(x_j)_i$ and such that the equation $1_z=a$ has a solution $x$. Then if one of the "distributive" laws $ab_i0=(ab0)_i$, $a_jc0=(ac0)_j$, holds, the other holds also.*

*Proof.* Suppose $ab_i0=(ab0)_i$ for all $a$, $b$, $i$, and $1_i=b$. Then

$$a_jb0 = a_j1_i0 = (a_j10)_i = (a_j)_i = (a_i)_j = (a1_i0)_j = (ab0)_j.$$

Next, suppose $a_i b0 = (ab0)_i$ for all $a$, $b$, $i$, and let $a = 1_i$. Then

$$ab_j0 = 1_i b_j 0 = (1b_j0)_i = (b_j)_i = (b_i)_j = (1_i b0)_j = (ab0)_j.$$

Next consider an $H$-system $H(A)$ in which the "associative" law

$$1(1xy)z = 1x(1yz) \tag{5}$$

holds for all elements $x$, $y$, $z$.

Define a new operation $+$ by

$$x + y = 1xy.$$

It is easily seen that $H(A)$ forms a group under $+$. Also the order of any element $x$ under $+$ is the same as the $A$-order of $x$. The following theorems follow at once.

THEOREM 4.7. *$N$, the number of elements in $H(A)$ is a multiple of the order of each element.*

THEOREM 4.8. *In order that there exist a finite plane geometry with $r+1$ points on a line, where $r$ is not a prime-power, it is necessary that the corresponding $H$-system be either non-"associative" (5) or non-"distributive."*

Let $H(A, D)$ be an $H$-system for which the associative law (5) and the distributive law (a) of Theorem 4.4 hold. From Theorems 4.2 and 4.3 we have for an $H(A, D)$ that $1x_i x_j = x_{i+j} = 1x_j x_i$ and $(x_i)_j = x_{ij}$.

THEOREM 4.9. *For an $H(A, D)$ the law $1_i 1_j 0 = 1_j 1_i 0$ holds.*

*Proof.* $1_i 1_j 0 = (1_i 10)_j = (1_i)_j = 1_{ij} = 1_{ji} = 1_j 1_i 0$.

THEOREM 4.10. *For an $H(A, D)$ the law $1_i(1_j 1_k 0)0 = (1_i 1_j 0)1_k 0$ holds.*

*Proof.* $1_i(1_j 1_k 0)0 = 1_i(1_{jk})0 = 1_{ijk} = 1_{ij}1_k 0 = (1_i 1_j 0)1_k 0$.

THEOREM 4.11. *Let an $H(A, D)$ be given such that for any element $a$ the equation $1_x = a$ has a solution $x$. Then the system forms a field under the two binary operations defined by $a+b = 1ab$, $ab = ab0$.*

*Proof.* That the system forms a group under addition has been proved. That this group is commutative follows since $11_i 1_j = 1_{i+j} = 11_j 1_i$ from Theorem 4.2. Multiplication $ab$ is commutative from Theorem 4.9. That multiplication is associative follows from Theorem 4.10. Finally $a(b+c) = ab+ac$ since for $a = 1_i$, $b = 1_j$, $c = 1_k$ we have

$$\begin{aligned} a(b + c) &= a(1bc)0 = 1_i(11_j 1_k)0 = 1_i(1_{j+k})0 = 1_{i(j+k)} \\ &= 1_{ij+ik} = 1(1_{ij})(1_{ik}) = 1(1_i 1_j 0)(1_i 1_k 0) \\ &= 1(ab0)(ac0) = ab + ac. \end{aligned}$$

THEOREM 4.12. *In order that an H-system with a prime number p of elements define a field under the binary operations of Theorem* 4.11 *it is necessary and sufficient that the H-system be an H(A, D).*

*Proof.* (Necessity). If the two operations define a field the associative law $1(1xy)z = 1x(1yz)$ holds since $(x+y)+z = x+(y+z)$. Since $x_i = 1x_{i-1}x = x_{i-1}+x$, we have

$$x_i = \overbrace{x + \cdots + x}^{i}$$

and $ax,0 = ax_i = (ax)_i = (ax0)_i$ and the $H$-system is distributive.

(Sufficiency). If the $H$-system is an $H(A, D)$, then all elements including 1 have orders equal to $p$, and by Theorem 4.11 a field is obtained.

It is of interest to note that if an $H$-system with a prime number of elements defines the double binary system above such that all of the field postulates except the two commutative laws are satisfied then these two laws follow from the others.

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# A SET OF TRIPLY PERSPECTIVE TRIANGLES ASSOCIATED WITH PROJECTIVE TRIADS

F. G. STOCKTON, Shell Development Company

**1. Notation.** The letters $A$, $B$, and $C$ will refer to triangles, upper case being used for general reference or to direct attention to the points of a triangle, lower case when the lines are of primary interest. Subscripts refer to individual points or lines. Thus the points of triangle $A$ are $A_1$, $A_2$, $A_3$, and the lines of $a$, the same triangle, are $a_1$, $a_2$, $a_3$; $a_1$ is the line not on $A_1$, *etc.* When homogenous coordinates are used the coordinates of $A_2$ will be $(A_{21}, A_{22}, A_{23})$, *etc.*

The letters $l$, $m$, $n$, with subscripts, will denote certain other lines needed in the development. The letters $r$, $s$, $t$, $u$, with subscripts, will represent numbers.

**2. Labelling the configuration of Pappus.** Take two distinct points $A_1$ and $A_2$ in the projective plane, and assume six distinct lines; $l_1$, $m_1$, $n_1$ on $A_1$ but not on $A_2$; $l_2$, $m_2$, $n_2$ on $A_2$ but not on $A_1$. Then the projectivity $l_1m_1n_1 \sim l_2m_2n_2$ determines a configuration of Pappus whose nine points and nine lines may be enumerated and labelled as follows (Fig. 1):

| | |
|---|---|
| $A_1$ = given | $l_1$ = given |
| $A_2$ = given | $l_2$ = given |
| $B_1 = l_1n_2$ | $l_3 = B_2C_3$ |
| $B_2 = l_2m_1$ | $m_1$ = given |
| $B_3 = m_2n_1$ | $m_2$ = given |
| $C_1 = l_1m_2$ | $m_3 = B_1C_2$ |
| $C_2 = l_2n_1$ | $n_1$ = given |
| $C_3 = m_1n_2$ | $n_2$ = given |
| | $n_3 = B_2C_1$ |

$A_3 = l_3m_3n_3$ = the Pappus Point of the projectivity.

If the projectivity is not a perspectivity, then $A_1$, $A_2$, and $A_3$ are not collinear, that is, they are the points of a non-singular triangle. In this case it may happen that $B_1$, $B_2$, $B_3$ are collinear, or that $C_1$, $C_2$, $C_3$ are collinear, but not both. It is assumed in much of the following work that $A$, $B$, and $C$ are non-singular triangles.

It has been remarked,* and reference to the construction will demonstrate, that $A$ is perspective with $B$ from three centers, namely the three points of $C$. This is a circular relation; $A$ and $C$ are triply perspective from the three points of $B$, and $B$ and $C$ are triply perspective from the three points of $A$.

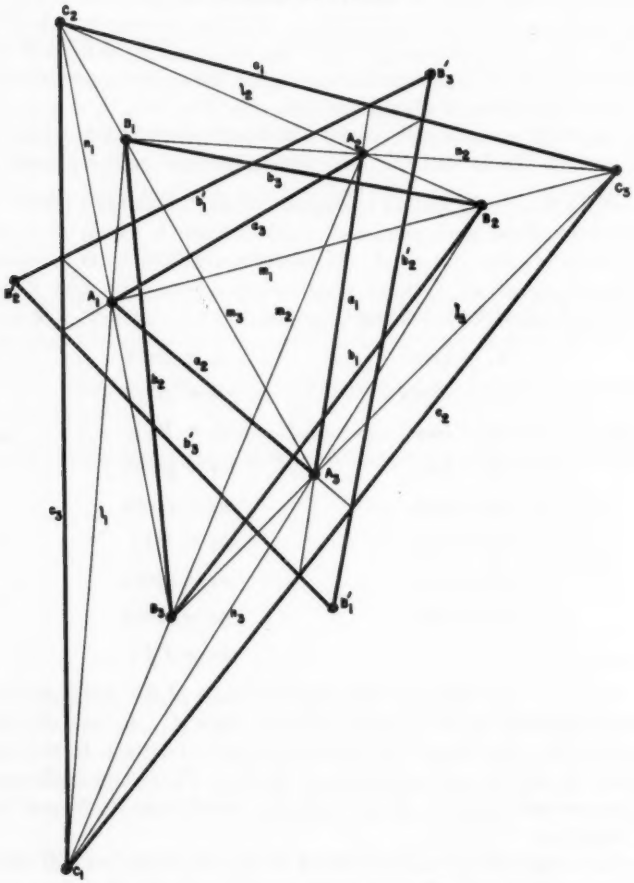
* O. Veblen and J. W. Young, Projective Geometry, Vol. 1, 1910, p. 100.

More explicitly, with the labelling chosen, any pair of the three triangles are cyclically centrally perspective in *reverse order*. That is:

$$A_1A_2A_3 \overset{C_2}{\wedge} B_3B_2B_1$$

$$A_1A_2A_3 \overset{C_3}{\wedge} B_2B_1B_3$$

$$A_1A_2A_3 \overset{C_1}{\wedge} B_1B_3B_2$$



$$C$$

FIG. 1. Labelling system for the Central Cell $B\ A$ (A Configuration of Pappus) and for the Axial Cell $C\ B'$

$$A$$

and similarly for the other pairs. The center of perspective bears the subscript of the two points which correspond in the perspectivity and have the same subscript.

**3. Definitions.**

*Cell.* Three triangles which are, by pairs, triply perspective in reverse order from the points of the third triangle will be called a *central cell of triply perspective triangles.* The dual figure will be called an *axial cell.* A cell is *non-singular* if all three triangles of the cell are non-singular. If one of the triangles of a central cell is singular (three distinct but collinear points) the cell is a *singular central cell.* The dual figure is a *singular axial cell.*

*Resultant.* Any triangle of a central cell is the *central resultant* of the other two. Any triangle of an axial cell is the *axial resultant* of the other two.

*Lattice.* Consider a central cell of triply perspective triangles $A$, $B$, and $C$ (Fig. 1). By Desargues' triangle theorem triangles $a$ and $c$ are triply axially perspective on three lines. The lines constitute a new§ triangle, $b'$, and $b'$, $c$, and $a$ are the triangles of an axial cell. In the same way the axial resultant of $b$ and $a$, and the axial resultant of $b$ and $c$ can be constructed. The central resultant of $C$ and $B'$ might be constructed next, and the process can be indefinitely continued.

The totality of triangles derivable from two triply perspective triangles by such repeated use of Desargues' triangle theorem and its dual will be called the *lattice of triangles.*

*Cycle.* The set of triangles derivable from two triply perspective triangles, $A$ and $B$, by alternate formation of central and axial resultants, $A$ being always paired with the last triangle formed, will be called a *cycle of triangles around* $A$. The cycle around $A$ is a subset of the lattice. Each triangle in the cycle is triply perspective with $A$.

**4. Relations among triangles of a cycle.**

LEMMA 1. *The central resultant of two triangles of a non-singular axial cell is a non-singular triangle.*

Assume it is singular. Then the points of the paired triangles lie on a conic by the converse of Pascal's theorem; the lines of the axial resultant of the two triangles are the polars of the collinear points of the central resultant, hence concurrent, contrary to hypothesis.

LEMMA 2. *The homogeneous coordinates of the points of any triangle $B$, triply centrally perspective in reverse order with the reference triangle, and not singularly placed with respect to it, may be written in the form:*

$$B_1 = (\ 1, \quad r, \quad s)$$
$$B_2 = (\ t, \quad 1, \quad u)$$
$$B_3 = (rt/s, \quad rt/u, \quad 1),$$

§ The conditions allowing $b=b'$ are examined in a later paragraph.

*where $r$, $s$, $t$, and $u$ are in general any four numbers, but subject to certain restrictions to insure non-singular placement.*

The vanishing of any coordinate would indicate singular placement. Therefore the coordinates of $B_1$ and $B_2$ can be reduced to the form given, and $B_{33}$ can be made unity. Solution of the reverse triple perspectivity condition:

$$B_{31}s = B_{32}u = rt$$

for $B_{31}$ and $B_{32}$ completes the proof.

THEOREM 1. *Let $A$ and $B$ be non-singular triangles, triply perspective in reverse order, and let $C$ be their non-singular central resultant. It is assumed, in addition, that $A$ and $B$ are not singularly related, that is, no point of either is on a line of the other. Construct the cycle of triangles around $A$, beginning with $B$ ($B$, $C$, $B'$, $C'$, $B''$, $C''$, $B'''$, $C'''$, $\cdots$). Then $B'''\equiv B$, and the cycle contains only six distinct triangles.*

*Proof.* Take $A$ as the reference triangle. In view of Lemma 2 take:

$$B_1 = (\ 1, \quad r_0, \quad s_0)$$
$$B_2 = (\ t_0, \quad 1, \quad u_0)$$
$$B_3 = (r_0t_0/s_0, \quad r_0t_0/u_0, \quad 1).$$

Then

$$C_1 = \begin{pmatrix} A_1B_1 \\ A_3B_2 \end{pmatrix} = (r_0t_0, \quad r_0, \quad s_0\ )$$

$$C_2 = \begin{pmatrix} A_2B_2 \\ A_3B_1 \end{pmatrix} = (\ t_0, \quad r_0t_0, \quad u_0\ )$$

$$C_3 = \begin{pmatrix} A_1B_2 \\ A_2B_1 \end{pmatrix} = (\ u_0, \quad s_0, \quad s_0u_0).$$

The restrictions imposed on the numbers $r_0$, $s_0$, $t_0$, $u_0$, by the assumptions are now explicitly set forth.*

1. No point of $B$ is on a line of $A$.
   Therefore $r_0\neq 0$, $s_0\neq 0$, $t_0\neq 0$, $u_0\neq 0$.
2. No point of $A$ is on a line of $B$.
   Therefore $s_0-r_0u_0\neq 0$, $u_0-s_0t_0\neq 0$, $r_0t_0-1\neq 0$.

---

* The single statement, "Neither the matrices of triangles $B$ and $C$ nor any of their minors of any order are singular," is equivalent to the restrictions given.

3. $B$ is non-singular.
Therefore

$$1 + \frac{r_0^2 t_0 u_0}{s_0} + \frac{r_0 s_0 t_0^2}{u_0} - 3 r_0 t_0 \neq 0,$$

and in view of restriction 1, this may be reduced to:

$$\frac{1}{r_0 t_0} + \frac{r_0 u_0}{s_0} + \frac{s_0 t_0}{u_0} \neq 3.$$

4. $C$ is non-singular.
Therefore $r_0^2 s_0 t_0^2 u_0 + r_0 u_0^2 + s_0^2 t_0 - 3 r_0 s_0 t_0 u_0 \neq 0$, and in view of restriction 1, this may be reduced to:

$$\frac{r_0 t_0}{1} + \frac{s_0}{r_0 u_0} + \frac{u_0}{s_0 t_0} \neq 3.$$

The lines of triangle $c$ can be expressed in the canonical form of the dual of Lemma 2 in terms of four numbers $r_1, s_1, t_1, u_1$. By expressing them also in terms of $r_0, s_0, t_0, u_0$, it can be shown that

$$r_1 = \frac{u_0 - s_0 t_0}{s_0(r_0 t_0 - 1)}$$

$$s_1 = \frac{t_0(s_0 - r_0 u_0)}{s_0 u_0(r_0 t_0 - 1)}$$

$$t_1 = \frac{s_0 - r_0 u_0}{u_0(r_0 t_0 - 1)}$$

$$u_1 = \frac{r_0(u_0 - s_0 t_0)}{s_0 u_0(r_0 t_0 - 1)}.$$

The new numbers $r_1, s_1, t_1, u_1$ satisfy restriction 1 since the first set satisfies restrictions 1 and 2.

They satisfy restriction 2 since the first set satisfies restriction 4.

They satisfy restriction 3 since triangle $c$ is non-singular.

They satisfy restriction 4 since $b'$ is non-singular, which is true by the dual of Lemma 1, $A$, $C$, and $B$ being non-singular.

Since $B'$ is triply perspective with $A$ in reverse order, the coordinates of its points can be expressed in the canonical form of Lemma 2 in terms of four numbers $r_2, s_2, t_2, u_2$. It can then be shown that

$$r_2 = \frac{u_1 - s_1 t_1}{s_1(r_1 t_1 - 1)}$$

$$s_2 = \frac{t_1(s_1 - r_1 u_1)}{s_1 u_1(r_1 t_1 - 1)}$$

$$t_2 = \frac{s_1 - r_1 u_1}{u_1(r_1 t_1 - 1)}$$

$$u_2 = \frac{r_1(u_1 - s_1 t_1)}{s_1 u_1(r_1 t_1 - 1)}$$

and it can be demonstrated as before that the new number set satisfies the four restrictions.

Since $B''$ is related to $B'$, and $B'''$ to $B''$, in precisely the same way that $B'$ is related to $B$:

$$B_1'' = (\ 1, \quad r_6, \quad s_6)$$

$$B_2'' = (\ t_6, \quad 1, \quad u_6)$$

$$B_3''' = (r_6 t_6/s_6, \ r_6 t_6/u_6, \quad 1\ ),$$

where $r_6$, $s_6$, $t_6$, $u_6$ are derived from $r_0$, $s_0$, $t_0$, $u_0$ by the recursion formulas:

$$r_{i+1} = \frac{u_i - s_i t_i}{s_i(r_i t_i - 1)}$$

$$s_{i+1} = \frac{t_i(s_i - r_i u_i)}{s_i u_i(r_i t_i - 1)}$$

$$t_{i+1} = \frac{s_i - r_i u_i}{u_i(r_i t_i - 1)}$$

$$u_{i+1} = \frac{r_i(u_i - s_i t_i)}{s_i u_i(r_i t_i - 1)}.$$

Expansion of the recursion formulas for $i+2$ gives:

$$r_{i+2} = 1/s_{i+1} u_i$$

$$s_{i+2} = 1/u_{i+1} t_i$$

$$t_{i+2} = 1/u_{i+1} s_i$$

$$u_{i+2} = 1/s_{i+1} r_i.$$

Armed with these last formulas, the following table can be constructed from left to right:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $r$ | $r_0$ | $r_1$ | $1/s_1u_0$ | $t_0$ | $t_1$ | $1/u_1s_0$ | $r_0$ |
| $s$ | $s_0$ | $s_1$ | $1/u_1t_0$ | $s_1r_0/t_1$ | $r_1s_0/t_0$ | $1/t_1u_0$ | $s_0$ |
| $t$ | $t_0$ | $t_1$ | $1/u_1s_0$ | $r_0$ | $r_1$ | $1/s_1u_0$ | $t_0$ |
| $u$ | $u_0$ | $u_1$ | $1/s_1r_0$ | $u_1t_0/r_1$ | $t_1u_0/r_0$ | $1/r_1s_0$ | $u_0$ |

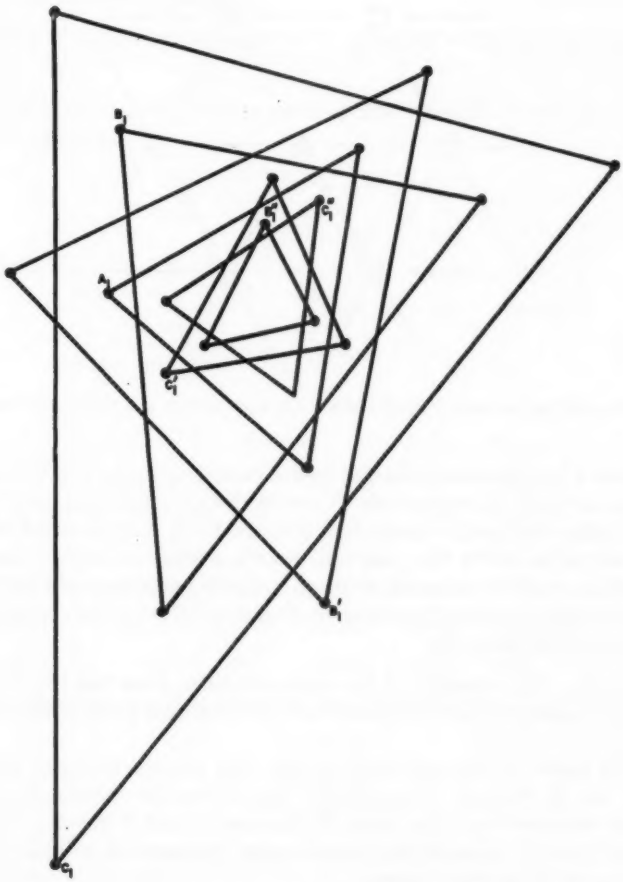



FIG. 2. The Cycle Around $A$.

Therefore $B''' \equiv B$, since the points of each have the same coordinates.

*Discussion.* Theorem 1 is a theorem of closure, establishing the cycle of triangles around $A$ as a closed set of six triangles. Figure 2 shows the cycle around $A$ derived from triangles $A$ and $B$ of Fig. 1. To simplify the figure only the first point of each triangle is labelled. The other points are in clockwise order, and the lines are to be labelled as described under "Notation" above.

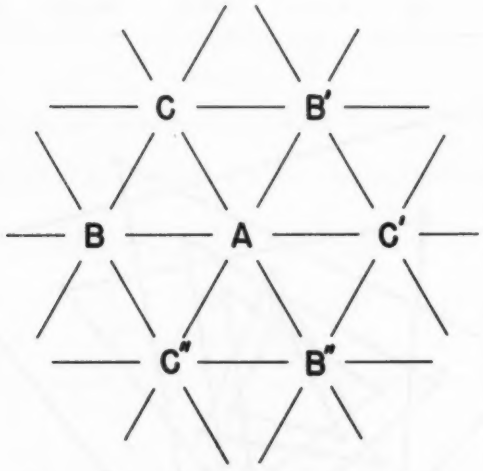



FIG. 3. Isomorphism between Triangles of a Cycle and Vertices of a Triangular Tesselation.

Theorem 1 implies a striking and useful isomorphism in which each triangle of the cycle around $A$ corresponds to one of the six points adjacent to a given point in a plane triangular tesselation (Figure 3). It will be noted that paired triangles are adjacent in the diagram, that a triangular unit of the diagram corresponds to a cell of triangles, that central cells point upwards and axial cells point downwards. Repeated recourse to Theorem 1 during the construction of a lattice of triangles yields the

COROLLARY. *The triangles of the lattice derivable from two triply perspective triangles are isomorphic with the vertices of an unbounded plane triangular tesselation.*

*Singular cycles.* It has not been shown that the six triangles of the cycle around $A$ are all distinct. The question can be readily examined using point coordinates obtained from the table of Theorem 1, and it develops that, under restrictions 1 and 2, none of the triangles can be identical with any of the six permuted forms of another unless:

$$\frac{r_0l_0}{1} = \frac{s_0}{r_0u_0} = \frac{u_0}{s_0l_0} = \sqrt[3]{1} = \omega \text{ or } \omega^2 \neq 1.$$

Under this constraint $B \equiv B' \equiv B''$ and $C \equiv C' \equiv C''$ so that the cycle around $A$ contains only triangles $B$ and $C$, and the lattice generated from $A$ and $B$ contains only $A$, $B$, and $C$. $A$, $B$, and $C$ are *all three* triply perspective from the points of a fourth non-singular triangle $D$. The twelve points of the four triangles form a configuration $\begin{matrix}12 & 3\\ 4 & 9\end{matrix}$. Any two of the four triangles are in sextuple perspective from the six points of the other two.*

**5. Pairing unpaired triangles.**

THEOREM 2. *Each two triangles in a cycle are triply perspective in reverse order.*

It is sufficient to prove the theorem for the pairs $B$, $B'$ and $b$, $c'$. From the table of Theorem 1:

$$B' = \begin{pmatrix} 1 & \dfrac{1}{s_1u_0} & \dfrac{1}{u_1l_0} \\ \dfrac{1}{u_1s_0} & 1 & \dfrac{1}{s_1r_0} \\ \dfrac{l_0}{s_0s_1u_0} & \dfrac{r_0}{u_1s_0u_0} & 1 \end{pmatrix} = \begin{pmatrix} s_1l_0u_0u_1 & l_0u_1 & s_1u_0 \\ r_0s_1 & r_0s_0s_1u_1 & s_0u_1 \\ l_0u_1 & r_0s_1 & s_0s_1u_0u_1 \end{pmatrix}.$$

Since

$$B = \begin{pmatrix} 1 & r_0 & s_0 \\ l_0 & 1 & u_0 \\ \dfrac{r_0l_0}{s_0} & \dfrac{r_0l_0}{u_0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & r_0 & s_0 \\ l_0 & 1 & u_0 \\ r_0 & \dfrac{r_0s_0}{u_0} & \dfrac{s_0}{l_0} \end{pmatrix},$$

$$\begin{vmatrix} [B_1B_3'] \\ [B_2B_2'] \\ [B_3B_1'] \end{vmatrix} = \begin{vmatrix} r_0s_0s_1u_0u_1 - r_0s_0s_1 & s_0l_0u_1 - s_0s_1u_0u_1 & r_0s_1 - r_0l_0u_1 \\ s_0u_1 - r_0s_0s_1u_0u_1 & r_0s_1u_0 - s_0l_0u_1 & r_0s_0s_1l_0u_1 - r_0s_1 \\ r_0s_0s_1 - s_0u_1 & s_0s_1u_0u_1 - r_0s_1u_0 & r_0l_0u_1 - r_0s_0s_1l_0u_1 \end{vmatrix}$$

is singular since the sum of the rows is zero. The determinant of the three lines

$$\begin{vmatrix} [B_1B_2'] \\ [B_2B_1'] \\ [B_3B_3'] \end{vmatrix}$$

* W. R. Andress and W. Saddler, Perspective triads. With a note by W. W. Sawyer. Math. Gaz. vol. 37, 1953, pp. 247–255.

is also singular. Therefore

$$B_1B_2B_3 \barwedge B_3' B_2' B_1'$$

and

$$B_1B_2B_3 \barwedge B_2' B_1' B_3',$$

so that triangles $B$ and $B'$ are doubly (and hence triply)* perspective in reverse order.

The reverse triple perspectivity of $b$ with $c'$ is demonstrable in the same way.

**6. Recursive identities.** Alternative proofs of Theorem 1 exist, each depending on a permutation of the coordinates of $B_1$ and $B_2$ in Lemma 2. Each such proof requires the development of a recursive identity in four numbers. In this way six distinct identities are encountered; they are recorded in Table 1 below, together with their inverses. The identities may be interpreted as transformations of period six in a four dimensional vector space.

Any of the identities may be written:

$$V_6 \equiv V_0$$

where $V_i$ is the vector $(r_i, s_i, t_i, u_i)$,

$$r_{i+1} = f_r(r_i, s_i, t_i, u_i),$$
$$s_{i+1} = f_s(r_i, s_i, t_i, u_i),$$
$$t_{i+1} = f_t(r_i, s_i, t_i, u_i),$$
$$u_{i+1} = f_u(r_i, s_i, t_i, u_i),$$

and $f_r, f_s, f_t, f_u$ are defined for each identity as tabulated below (Table 1).

The identities hold for any initial vector $V_0$ satisfying four mild restrictions. There are two such sets of restrictions, I and II. Restrictions I are those given in Theorem 1; restrictions II are listed below. Each identity is governed by one set or the other, as noted in Table 1.

*Restrictions II*

1. $r_0 \neq 0,\ s_0 \neq 0,\ t_0 \neq 0,\ u_0 \neq 0$
2. $r_0u_0 - s_0t_0 \neq 0,\ t_0 - r_0 \neq 0,\ s_0 - u_0 \neq 0$
3. $\dfrac{s_0t_0}{r_0u_0} + \dfrac{r_0}{t_0} + \dfrac{u_0}{s_0} \neq 3$
4. $\dfrac{r_0u_0}{s_0t_0} + \dfrac{t_0}{r_0} + \dfrac{s_0}{u_0} \neq 3.$

* O. Veblen and J. W. Young, *op. cit.*, p. 247.

*Illustration.* The following small table gives the successive values of $r_i$, $s_i$, $t_i$, $u_i$ for the identity $J_1$ and the initial vector $V_0 = (1, 3, 2, 9)$. The $J_1'$ identity gives the same intermediates in the reverse order.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $r$ | 1 | 1 | −1/4 | 2 | −2/3 | 3 | 1 |
| $s$ | 3 | −4/9 | 9/2 | 2/3 | 3/2 | −1/6 | 3 |
| $t$ | 2 | −2/3 | 3 | 1 | 1 | −1/4 | 2 |
| $u$ | 9 | 1/9 | −9/4 | 2/9 | −6 | 1/3 | 9 |

*Acknowledgement*

The author is indebted to Dr. T. K. Pan, Department of Mathematics, University of Oklahoma, for helpful criticism and encouragement.

TABLE 1. $f_r$, $f_s$, $f_t$, AND $f_u$ FOR THE SIX IDENTITIES AND THEIR INVERSES

| Identity | $f_r$ | $f_s$ | $f_t$ | $f_u$ | Restrictions |
|---|---|---|---|---|---|
| $J_1$ | $\frac{u-st}{s(rt-1)}$ | $\frac{t(s-ru)}{su(rt-1)}$ | $\frac{s-ru}{u(rt-1)}$ | $\frac{r(u-st)}{su(rt-1)}$ | I |
| $J_1'$ | $\frac{u(rt-1)}{r(u-st)}$ | $\frac{rt-1}{s-ru}$ | $\frac{s(rt-1)}{t(s-ru)}$ | $\frac{rt-1}{u-st}$ | I |
| $J_2$ | $\frac{t(s-ru)}{r(u-st)}$ | $\frac{rt-1}{r(u-st)}$ | $\frac{r(s-ru)}{s(rt-1)}$ | $\frac{r(u-st)}{su(rt-1)}$ | I |
| $J_2'$ | $\frac{u-st}{s-ru}$ | $\frac{u-st}{su(rt-1)}$ | $\frac{ru(rt-1)}{t(s-ru)}$ | $\frac{rt-1}{u-st}$ | I |
| $J_3$ | $\frac{u(rt-1)}{s-ru}$ | $\frac{r(u-st)}{s(s-ru)}$ | $\frac{r(u-st)}{t(s-ru)}$ | $\frac{r(rt-1)}{s-ru}$ | I |
| $J_3'$ | $\frac{t(s-ru)}{s(rt-1)}$ | $\frac{t(s-ru)}{s(u-st)}$ | $\frac{s-ru}{u-st}$ | $\frac{rt(s-ru)}{su(rt-1)}$ | I |
| $J_4$ | $\frac{s(rt-1)}{r(s-ru)}$ | $\frac{u-st}{u(s-ru)}$ | $\frac{u(rt-1)}{t(u-st)}$ | $\frac{s-ru}{s(u-st)}$ | I |
| $J_4'$ | $\frac{t(s-ru)}{ru(rt-1)}$ | $\frac{t(s-ru)}{ru(u-st)}$ | $\frac{r(u-st)}{st(rt-1)}$ | $\frac{r(u-st)}{st(s-ru)}$ | I |
| $J_5$ | $\frac{s-u}{s(t-r)}$ | $\frac{ru-st}{su(t-r)}$ | $\frac{u(t-r)}{t(ru-st)}$ | $\frac{r(s-u)}{s(ru-st)}$ | II |
| $J_5'$ | $\frac{u(t-r)}{rt(s-u)}$ | $\frac{t-r}{ru-st}$ | $\frac{ru-st}{st(t-r)}$ | $\frac{ru-st}{st(s-u)}$ | II |
| $J_6$ | $\frac{ru-st}{rt(s-u)}$ | $\frac{t-r}{t(s-u)}$ | $\frac{s-u}{s(t-r)}$ | $\frac{ru-st}{su(t-r)}$ | II |
| $J_6'$ | $\frac{s-u}{ru-st}$ | $\frac{r(s-u)}{su(t-r)}$ | $\frac{u(t-r)}{rt(s-u)}$ | $\frac{t-r}{ru-st}$ | II |

# AN AXIOMATIC TRIANGULAR GEOMETRY

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**1. Introduction.** The analytic trilinear geometry of the plane, in which the (three) coordinates of a point are defined as proportional to its distances from the sides of a fixed triangle, or to the areas it determines with these sides, has accumulated quite a literature, especially in the decades following 1860 [1]. These geometries, however, do not seem to have received any systematic axiomatic treatment. It is the purpose of this paper to present a set of axioms for such a geometry in the plane (readily generalizable to $n$ dimensions), indicating in the process some interesting definitions and concepts and illustrating the power of the axioms by developing some specific branches of the theory.

**2. The axioms.** Suppose given a set of (undefined) elements, which we shall call "points," and denote by capital letters. Let there further be given a function mapping the set of ordered triples of points into the reals, denoted by $\Delta(A, B, C)$, read "area (of the triangle determined by) $A$, $B$, $C$."

We now assume that $\Delta$ satisfies the following axioms:

(S) Symmetry axiom:

$$\Delta(\pi(A, B, C)) = (\mathrm{sg}(\pi))\cdot\Delta(A, B, C)$$

(*where $\pi$ is a permutation*).

(I) Identity axiom: *If*

$$\sum_{i=1}^{n} \alpha_i\cdot\Delta(X, A_i, B_i) = \alpha_0 \qquad (\alpha_i \textit{ reals})$$

*is satisfied by $X_1$, $X_2$, $X_3$ with $\Delta(X_1, X_2, X_3)\neq 0$, then it is an identity in $X$.*

(E) Existence axiom: *If $A$, $B$ are distinct points, there exist points $C$, $D$ such that*

$$\Delta(A, B, C) \neq \Delta(A, B, D).$$

*Indeed, given any such $A$, $B$, $C$, $D$, and any pair $a$, $b$ of reals, there exists a point $X$ such that*

$$\Delta(X, A, B) = a; \qquad \Delta(X, C, D) = b.$$

**3. Collinearity.** We proceed to investigate the structure given to our set of points by these axioms, beginning by defining the line.

DEFINITION: "*$A$, $B$, $C$ are collinear*" *means* $\Delta(A, B, C)=0$.

DEFINITION: *The line determined by $A$, $B$ is the set of $X$ such that* $\Delta(A, B, X)=0$.

We may now show collinearity to have the expected properties:

(C1) Reflexivity: *By* (S), $\Delta(A, B, C)=0$ *if not all of $A$, $B$, $C$ are distinct.*

(C2) Symmetry: *By* (S), $\Delta(A, B, C) = 0$ *implies* $\Delta(\pi(A, B, C)) = 0$.

(C3) Transitivity: *If* $A, B, C$ *are distinct, and* $\Delta(A, B, C) = \Delta(B, C, D) = 0$, *then* (*e.g.*) $\Delta(A, C, D) = 0$; for, if not, since $A, C, D$ by hypothesis all satisfy $\Delta(X, B, C) = 0$, by (I) so would any point $X$, contrary to (*E*).

(We remark in passing that the properties (C1–C3) of the ternary relation "collinearity" are quite analogous to those defining the familiar (binary) *equivalence relation*. The generalization to $n$-ary relations is quite straightforward.)

By means of suitable definitions it is now not difficult to establish most of the *Euclidean* axioms on united position of points and lines, the order of points on a line, *etc.*, whence to define *ray, line segment, etc.* However, we shall not have occasion to make much use of such results in the sequel.

**4. Metric properties of Δ.** It may further be observed that our Δ-function (in common with its natural realization in the Euclidean plane—the area of a triangle) has properties somewhat analogous to those which define a *distance*-function or *metric*. In particular, we have the

(T) Triangle equality:

$$\Delta(A, B, C) = \Delta(X, B, C) + \Delta(A, X, C) + \Delta(A, B, X).$$

*Proof*: If $\Delta(A, B, C) \neq 0$, then since $X = A, B, C$ satisfy (T), so does any $X$, by (I). If $\Delta(A, B, C) = 0$, either all of $\Delta(X, B, C)$, $\Delta(A, X, C)$, $\Delta(A, B, X)$ are zero (Q.E.D.), or (by C3) none are zero. (The case where $A, B, C$ are not all distinct may be handled by using (C1) and (S).) But if (*e.g.*) $\Delta(A, B, X) \neq 0$, our first argument proves

$$\Delta(A, B, X) = \Delta(C, B, X) + \Delta(A, C, X) + \Delta(A, B, C)$$

which by (S) is readily equivalent to (T).

(The similarity between (C1, S, T) and the defining properties of a *metric* is strengthened by the following observations: In our *Euclidean* model, if we pass to 3-space, (T) does indeed take on the more familiar form of an *inequality*, expressible in words as: *The area of any face of a tetrahedron does not exceed the sum of the areas of the other three faces*. The plane case for (signed) "area" is thus the analogue of the *line* case for (signed) "distance," as may be immediately seen.)

**5. Coordinates.** Axiom (I), together with the results already established, enables us to set up formal *triangular coordinates* on our set of points. Indeed, we could actually define general *two-segment* coordinates (a system in which the coordinates of $X$ with respect to the ordered point-pairs $A, B$; $C, D$ are defined as the reals of axiom (E)); but we choose for simplicity to let the two segments have an end-point in common, making them $(A, B)$ and $(A, C)$, say. For this case it is readily seen that the nondegeneracy hypothesis of (E) reduces to the requirement that $A, B, C$ be noncollinear. For purposes of homogeneity we further choose to define as our coordinates the *ratios*

$$X_B \equiv \frac{\Delta(A, X, C)}{\Delta(A, B, C)}, \qquad X_C \equiv \frac{\Delta(A, B, X)}{\Delta(A, B, C)}$$

(permissible since we are assuming $\Delta(A, B, C) \neq 0$). In the Euclidean realization of our system these may now easily be recognized as two of the familiar areal triangular coordinates of $X$. For symmetry we may, finally, think of our coordinates as related to the "basic *triple*" of points $A$, $B$, $C$, introducing the third coordinate

$$X_A = \frac{\Delta(X, B, C)}{\Delta(A, B, C)}.$$

By (T) we have evidently

$$X_A + X_B + X_C = 1.$$

It remains only to complete the one-to-one correspondence between points and pairs of coordinates by proving the following

THEOREM. (U) Uniqueness: *With respect to a given basic triple $A$, $B$, $C$, a point is determined uniquely by two of its coordinates, i.e., $X_B = Y_B$ and $X_C = Y_C$ imply $X = Y$.*

*Proof*: If $X \neq Y$, then since $\Delta(A, B, C) \neq 0$, by (C3) at least one of $\Delta(A, X, Y)$' $\Delta(C, X, Y)$ is $\neq 0$; say $\Delta(A, X, Y)$. Similarly not both $\Delta(X, A, B)$, $\Delta(X, A, C)$ are zero; say $\Delta(X, A, B) \neq 0$. Suppose now that $\Delta(X, A, B) = \Delta(Y, A, B)$ and $\Delta(X, A, C) = \Delta(Y, A, C)$ (obviously equivalent to the hypothesis). Let $\Delta(X, A, C) = \lambda \cdot \Delta(X, A, B)$; then $\Delta(Y, A, C) = \lambda \cdot \Delta(Y, A, B)$. But also $0 = \lambda \cdot \Delta(A, A, B) = \Delta(A, A, C)$; and these three roots $X$, $Y$, $A$ are not collinear. Hence by (I), $\Delta(Z, A, C) = \lambda \cdot \Delta(Z, A, B)$ identically in $Z$, and must be satisfied *e.g.*, by $B$, so that $\Delta(B, A, C) = \lambda \cdot \Delta(B, A, B) = 0$, contradicting $\Delta(A, B, C) \neq 0$. Thus $X \neq Y$ and the hypotheses are incompatible.

**6. Transformations.** If we now change over from the basic triple $A$, $B$, $C$ to another, say $P$, $R$, $S$ (with $\Delta(P, R, S) \neq 0$), it is natural to inquire whether our system as thus far developed enables us to write $X_P$, $X_R$, $X_S$ in terms of $X_A$, $X_B$, $X_C$. We shall show that this is indeed the case, first proving:

(H) Homogeneity: *If* $\Delta(A, O, X) = \lambda \cdot \Delta(A, O, Y)$, *and* $\Delta(B, O, X) = \lambda \cdot \Delta(B, O, Y)$, *where* $\Delta(A, B, O) \neq 0$, *then for all* $P$, $\Delta(P, O, X) = \lambda \cdot \Delta(P, O, Y)$.

(A) Additivity: *If* $\Delta(A, O, X) = \Delta(A, O, Y) + \Delta(A, O, Z)$ *and* $\Delta(B, O, X) = \Delta(B, O, Y) + \Delta(B, O, Z)$, *where* $\Delta(A, B, O) \neq 0$, *then for all* $P$, $\Delta(P, O, X) = \Delta(P, O, Y) + \Delta(P, O, Z)$.

(These are immediate specializations of (I).)

(D) Determinant Theorem:

$$\frac{\Delta(X,Y,Z)}{\Delta(A,B,C)} = \begin{vmatrix} X_A & X_B & X_C \\ Y_A & Y_B & Y_C \\ Z_A & Z_B & Z_C \end{vmatrix}.$$

*Proof*: By (T), this determinant is equal to (*e.g.*)

$$\begin{vmatrix} 1 & X_B & X_C \\ 1 & Y_B & Y_C \\ 1 & Z_B & Z_C \end{vmatrix}.$$

Consider its expansion by minors of its first column. If we can show that

$$\begin{vmatrix} Y_B & Y_C \\ Z_B & Z_C \end{vmatrix} = \frac{\Delta(A,Y,Z)}{\Delta(A,B,C)},$$

and similarly that

$$\begin{vmatrix} Z_B & Z_C \\ X_B & X_C \end{vmatrix} = \frac{\Delta(A,Z,X)}{\Delta(A,B,C)}; \qquad \begin{vmatrix} X_B & X_C \\ Y_B & Y_C \end{vmatrix} = \frac{\Delta(A,X,Y)}{\Delta(A,B,C)},$$

our theorem will follow by (S) and (T). We proceed to do so.

Since by (U) the points $Y$, $Z$ are uniquely determined by any two of the coordinates of each, evidently

$$f(Y,Z) \equiv \frac{\Delta(A,Y,Z)}{\Delta(A,B,C)}$$

*is* a function of $Y_B$, $Y_C$; $Z_B$, $Z_C$. This function has the following properties:

1. If $Y_B = Z_B$, $Y_C = Z_C$, then by (U), $Y = Z$, whence (by (C1)) $f(Y, Z) = 0$.
2. If $(Y_B, Y_C) = (1, 0)$; $(Z_B, Z_C) = (0, 1)$, by (U) evidently $Y = B$, $Z = C$, so that
$$f(Y,Z) = f(B,C) \equiv \frac{\Delta(A,B,C)}{\Delta(A,B,C)} = 1.$$
3. If $Y_B = \lambda \cdot Y_B'$, $Y_C = \lambda \cdot Y_C'$, by (H) we have $f(Y, Z) = \lambda \cdot f(Y', Z)$.
4. If $Y_B = Y_B' + Y_B''$, $Y_C = Y_C' + Y_C''$, by (A) we have $f(Y, Z) = f(Y', Z) + f(Y'', Z)$.

But (see, *e.g.*, [2]) a function of the two pairs $(Y_B, Y_C)$; $(Z_B, Z_C)$ which satisfies (1–4) must be their determinant.

(*N.B.*: The technique used in this proof is of interest in its own right, but its inherent limitations are illustrated by the remark that it could not have been used to prove the theorem directly in the 3-by-3 case, since the linearity requirements (3, 4) are meaningless when the sum of the entries in each row is necessarily 1.)

We may now apply (D) to answer the question posed at the beginning of this section, for we have

$$X_P \equiv \frac{\Delta(X, R, S)}{\Delta(P, R, S)} = \begin{vmatrix} X_A & X_B & X_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix} \Big/ \begin{vmatrix} P_A & P_B & P_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix},$$

and similarly for $X_R$ and $X_S$. In other words, $X_P$, $X_R$, $X_S$ are precisely the solutions of the system of linear equations

$$\begin{aligned} P_A X_P + R_A X_R + S_A X_S &= X_A \\ P_B X_P + R_B X_R + S_B X_S &= X_B \\ P_C X_P + R_C X_R + S_C X_S &= X_C, \end{aligned}$$

which solutions do indeed exist uniquely by virtue of

$$\Delta(P, R, S) = \Delta(A, B, C) \begin{vmatrix} P_A & P_B & P_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix} \neq 0.$$

This discussion gives us an analytic picture of our geometry. We may, indeed, now view $P$, $R$, $S$, not as a new basic triple, but as the image of $A$, $B$, $C$ under the coordinate transformation $(X_A, X_B, X_C) \rightarrow (X_P, X_R, X_S)$, which we have just seen to be linear. Indeed, since the property that the sum of the entries in each row is 1 (*barycentricity*) evidently passes to the matrix product and inverse, our geometry is actually isomorphic to the geometry defined by the group of (three-dimensional) linear transformations with barycentric matrices. And (D) incidentally tells us that such a transformation is "area"-preserving if and only if its matrix has determinant $+1$. Our system now has a well-defined place in the set of (numerical) "projective geometries."

**7. Further results.** A comparison of the "Euclidean" realization of our system with ordinary Euclidean geometry enables us to determine those concepts of the latter which may be generalized to the former. For example, since the medians of a triangle bisect its area, our coordinate system enables us to define "median" (for the basic triangle), whence (guided by the theorem on the line joining the midpoints of two sides of a triangle) the concept of congruence (of line segments) by collinear or parallel translation (see definition of "parallel" just below). However, "congruence" cannot be defined in this system for non-parallel segments. Indeed, if it could, we would then have in our model a general definition of "length" in terms of "area," so that any (*e.g.*, linear) transformation preserving area would also preserve length—which is of course not the case. In consequence of this, concepts like "circle," "isosceles triangle," *etc.*, whence "perpendicular," *etc.*, are undefinable within our system.

As an illustration of what *can* be done within our system as so far developed, we give the theory of parallels.

DEFINITION: *The point-pair $A$, $B$ "is parallel to" the point-pair $C$, $D$ (where $A \neq B$, $C \neq D$) if and only if* $\Delta(A, C, D) = \Delta(B, C, D)$.

We proceed to establish the following properties of "parallel":

(P1) *"Parallel" is well-defined for lines, indeed we have also* $\Delta(X, C, D) = \Delta(A, C, D)$ *if and only if* $\Delta(X, A, B) = 0$.

(N.B.: The "only if" part of this assertion is Playfair's Axiom.)

*Proof*: Take $C$, $D$ as two of the points of a basic triple $(P, C, D)$. By (D), we have

$$\frac{\Delta(X, A, B)}{\Delta(P, C, D)} = (e.g.) \begin{vmatrix} X_P & X_C & 1 \\ A_P & A_C & 1 \\ B_P & B_C & 1 \end{vmatrix} = \begin{vmatrix} X_P & 1 & X_D \\ A_P & 1 & A_D \\ B_P & 1 & B_D \end{vmatrix}.$$

But if $X_P = A_P = B_P$, *i.e.*,

$$\frac{\Delta(X, C, D)}{\Delta(P, C, D)} = \frac{\Delta(A, C, D)}{\Delta(P, C, D)} = \frac{\Delta(B, C, D)}{\Delta(P, C, D)},$$

this determinant is evidently zero. Conversely, if the determinant is zero, multiplying the last column by $A_P (= B_P)$, subtracting it from the first, and expanding by minors of the first column, gives

$$0 = \begin{vmatrix} X_P - A_P & X_C & 1 \\ 0 & A_C & 1 \\ 0 & B_C & 1 \end{vmatrix} = (X_P - A_P)(A_C - B_C).$$

Similar treatment of

$$\begin{vmatrix} X_P & 1 & X_D \\ A_P & 1 & A_D \\ B_P & 1 & B_D \end{vmatrix}$$

gives $0 = (X_P - A_P)(A_D - B_D)$. Hence if $X_P \neq A_P$, we would have $A_C = B_C$, $A_D = B_D$, or (by (U)) $A = B$, a contradiction. Thus $X_P = A_P$, which is obviously equivalent to our assertion.

(P2) *"Parallel" is equivalent to "identical or nonintersecting," i.e. if $C$, $D$ do not lie on the line determined by $A$, $B$, then this line is parallel to the line of $(C, D)$ if and only if it has no point in common with it.*

*Proof*: If these lines had the point $X$ in common, we would have (by (P1)) $0 = \Delta(X, C, D) = \Delta(A, C, D) = \Delta(B, C, D)$, making the lines identical by (C3), a contradiction. Conversely, if the lines were not parallel, by (E) there would

exist a point $X$ with $\Delta(X, A, B)=0$, $\Delta(X, C, D)=0$, *i.e.*, a point common to them, contrary to hypothesis.

(P3) *"Parallel" is reflexive*—by (C1).

(P4) *"Parallel" is symmetric*—by (T).

(P5) *"Parallel" is transitive, i.e. two lines parallel to a third line are parallel to each other*; for, if not, by (P2) they intersect (the case where they're identical is of course trivial), and an application of (P1) to their intersection-point shows that they must be identical.

We may proceed without much difficulty from this point to develop the theory of parallelograms, a theory which incidentally might also serve us as an alternative guide to the definition of congruence of parallel segments. The details are left to the interested reader.

**References**

1. Coolidge, A History of Geometrical Methods, Oxford, 1940, Book II, Chapter II, Section 2.
2. Artin, Galois Theory, Notre Dame, 1948, Chapter I, Section F.

# THREE MUTUALLY ORTHOGONAL REAL CIRCLES

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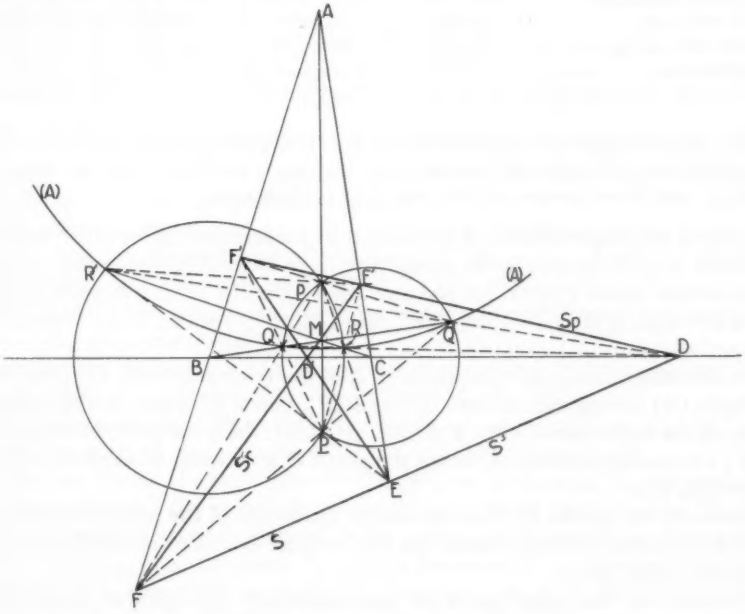
**1. Preliminaries.** The object of this paper is to consider the six points of intersection of three mutually orthogonal real circles $(A)$, $(B)$, $(C)$, chiefly in relation to the centers and axes of similitude, and to the circles of antisimilitude of those circles.

The triangle $ABC$ formed by the centers $A$, $B$, $C$ of the given circles will be said to be their central triangle.

The orthogonal center $M$ of the three circles considered necessarily lies inside those circles, hence their orthogonal circle $(M)$ is imaginary. We consider no points on the circumference of $(M)$, but we make some use of the polarity with respect to that circle [3].




**2. Triangles of intersection.** Let

(a) $$P, P'; Q, Q'; R, R'$$

be the three pairs of points of intersection of the three pairs of circles $(B)$, $(C)$; $(C)$, $(A)$; $(A)$, $(B)$, respectively. The two points of the same pair may be said to be complementary points of intersection.

The points located on the same circle are given by the following:

TABLE A

| | | | |
|---|---|---|---|
| $(A)$ | | $Q, Q';$ | $R, R'$ |
| $(B)$ | $P, P';$ | | $R, R'$ |
| $(C)$ | $P, P';$ | $Q, Q';$ | |

Taking one, and only one, point from each of the three pairs of points (a) we form eight triangles which may be referred to as the triangles of intersection of the three given circles. They are listed in the following:

TABLE B

| | I | II | III | IV |
|---|---|---|---|---|
| Triangles of intersection | $PQR, P'Q'R'$ | $PQ'R', P'QR$ | $P'QR', PQ'R$ | $P'Q'R, PQR'$ |
| Axes of similitude | $s=DEF$ | $s_p=DE'F'$ | $s_q=EF'D'$ | $s_r=FD'E'$ |
| Circles of intersection | $(I), (I')$ | $(I_p), (I_p')$ | $(I_q), (I_q')$ | $(I_r), (I_r')$ |
| Circumcenters | $I, I'$ | $I_p, I_p'$ | $I_q, I_q'$ | $I_r, I_r'$ |

The eight triangles of intersection are grouped into four pairs I, II, III, IV of complementary triangles of intersection. Two such triangles have no vertex in common, and thus involve all six points of intersection.

**3. Axes of perspectivity.** a. Any two complementary triangles of intersection, say, $PQR, P'Q'R'$ are obviously perspective from the orthogonal center $M$ of the given circles, hence their three pairs of corresponding sides intersect in three points $D=(QR, Q'R')$, $E=(RP, R'P')$, $F=(PQ, P'Q')$ which lie on their axis of perspectivity $s$.

On the other hand, the two points $Q, R$ of the two circles $(C)$, $(B)$ also lie on the circle $(A)$ orthogonal to both $(C)$ and $(B)$, hence $Q, R$ are antihomologous points on the latter two circles [1, p. 198, art. 435]; the same holds for the points $Q', R'$, for analogous reasons, hence the point $D$ is a center of similitude of the circles $(B)$, $(C)$.

Similarly the points $E, F$ are centers of similitude of the pairs of circles $(C)$, $(A)$; $(A)$, $(B)$, respectively. Hence the line $s=DEF$ is an axis of similitude of the circles $(A)$, $(B)$, $(C)$.

Likewise for the other pairs of complementary triangles of intersection. Thus: *The four axes of perspectivity of the four pairs of complementary triangles of intersection of three mutually orthogonal circles coincide with the four axes of similitude of these circles.*

The points $D'=(QR', Q'R)$, $E'=(RP', R'P)$, $F'=(PQ', P'Q)$ are the other three centers of similitude of the three pairs of circles $(B)$, $(C)$; $(C)$, $(A)$; $(A)$, $(B)$, respectively. The lines $s_p=DE'F'$, $s_q=EF'D'$, $s_r=FD'E'$ are the axes of perspectivity of the pairs of triangles II, III, IV, respectively (table B), and also the other three axes of similitude of the circles $(A)$, $(B)$, $(C)$.

b. The circumcircles of the triangles of intersection may conveniently be referred to as the circles of intersection of the circles $(A)$, $(B)$, $(C)$. Two circles of intersection will be said to be complementary if they are the circumcircles of two complementary triangles of intersection.

c. Let $(I)$, $(I')$ denote the circumcircles of the triangles $PQR$, $P'Q'R'$ (table B). The lines $DQR$, $DQ'R'$ (section 3a) are the radical axes of the pairs of circles $(I)$, $(A)$; $(I')$, $(A)$, respectively, hence the radical axis of $(I)$, $(I')$ passes through the point $D$. The points $E$, $F$ lie on that radical axis for analogous reasons. Thus the radical axis of $(I)$, $(I')$ coincides with the line $s = DEF$ (section 3a).

By a like argument it may be shown that the lines $s_p$, $s_q$, $s_r$ are the radical axes of the pairs of circles $(I_p)$, $(I_p')$; $(I_q)$, $(I_q')$; $(I_r)$, $(I_r')$ (table B). Thus: *The radical axis of two complementary circles of intersection* (section 3b) *of three mutually orthogonal circles coincides with the axis of perspectivity of the corresponding triangles of intersection.*

d. As an immediate consequence of sections 3a, 3c we have: *The radical axes of the four pairs of complementary circles of intersection of three mutually orthogonal circles coincide with the four axes of similitude of the latter three circles.*

**4. Circles of antisimilitude.** a. The point $D$ common to the radical axes of the three circles $(A)$, $(I)$, $(I')$ taken in pairs (section 3c) is the center of the orthogonal circle $(D)$ of those three circles.

On the other hand, the point $D$ is a center of similitude of the circles $(B)$, $(C)$ (section 3a), hence $D$ lies on the line $BC$. Moreover, the circle $(D)$ being orthogonal to $(A)$, the three circles $(B)$, $(C)$, $(D)$ are coaxial [1; p. 207, art. 458]. The circle $(D)$ thus passes through the points $P$, $P'$ and is a circle of antisimilitude of the given circles $(B)$, $(C)$.

It may be shown in a like manner that the point $D'$ is the center of the second circle of antisimilitude $(D')$ of the circles $(B)$, $(C)$.

Analogous considerations yield the circles of antisimilitude $(E)$, $(E')$; $(F)$, $(F')$ of the pairs of given circles $(C)$, $(A)$; $(A)$, $(B)$, respectively.

The two circles of antisimilitude $(D)$, $(D')$ are orthogonal to each other and both pass through the vertex $P$ of the triangle $PQR$; furthermore, $(D)$ is orthogonal to the circumcircle $(I)$ of $PQR$, as was shown at the beginning of the present paragraph, at the point $P$, and has its center $D$ on the side $QR$ opposite $P$ (section 3a). Hence the circle $(D')$ is tangent to $(I)$ at $P$, while the circle $(D)$ coincides with the Apollonian circle of the triangle $PQR$ passing through the vertex $P$.

For analogous reasons the circles of antisimilitude $(E)$, $(F)$ whose centers $E$, $F$ are collinear with $D$, on the axis $s$, are the Apollonian circles of the triangle $PQR$ passing through the vertices $Q$, $R$, respectively, and the circles $(E')$, $(F')$ are tangent to the circle $(I)$ at the vertices $Q$, $R$, respectively.

The arguments presented in connection with the triangle $PQR$ and its circumcircle $(I)$ are equally valid for the complementary triangle of intersection

$P'Q'R'$ and its circumcircle $(I')$, that is, the circles of antisimilitude $(D)$, $(E)$, $(F)$ are the Apollonian circles of $P'Q'R'$, and the circles $(D')$, $(E')$, $(F')$ touch the circle $(I')$ at the vertices $P'$, $Q'$, $R'$, respectively.

The rôle played by the axis of similitude $s = DEF$ with respect to the two complementary triangles of intersection of group I (table B) may be played by the axes of similitude $s_p$, $s_q$, $s_r$ with respect to the pairs of complementary triangles of intersection of the groups II, III, IV, respectively.

To sum up: *Given three mutually orthogonal real circles,* i. *Their three circles of antisimilitude, whose centers lie on any axis of similitude, are the Apollonian circles of each of the two complementary triangles of intersection whose axis of perspectivity coincides with the axis of similitude considered.*

ii. *Each of the remaining three circles of antisimilitude of the three given circles is tangent to the circumcircles of the two triangles of intersection considered, the point of contact being a vertex of the respective triangle.*

b. The preceding proposition (section 4a) gives rise to a considerable number of corollaries. Here are some:

i. *The axis of perspectivity of two complementary triangles of intersection is the Lemoine axis of each of the two triangles.*

ii. *Two complementary triangles of intersection have the same Brocard diameter and the same isodynamic points.*

iii. *The four Brocard diameters of the four pairs of complementary triangles of intersection are concurrent.*

Indeed, they have in common the orthogonal center $M$ of the three given circles.

iv. *The circumcircles of two complementary triangles of intersection have no points in common.*

Indeed, the two isodynamic points are inverse with respect to each of the two circumcircles $(I)$, $(I')$ [1; p. 262, art. 603], they are therefore the limiting points of the coaxial pencil determined by $(I)$, $(I')$ [1; p. 207, art. 457], hence the proposition.

c. The circle $(D)$ has its center $D = ss_p$ on the two axes of similitude $s$, $s_p$, hence $(D)$ is a circle of Apollonius for two, and only two, pairs of complementary triangles of intersection, namely, the triangles of the groups I and II, while the circle $(D')$ is tangent to the circumcircles of those two pairs of triangles. The rôles of the two circles $(D)$, $(D')$ are reversed for the triangles of the groups III and IV (section 4a).

Similarly for the other two pairs of circles of antisimilitude $(E)$, $(E')$; $(F)$, $(F')$. Thus: *Given three mutually orthogonal circles, each of their circles of antisimilitude is a circle of Apollonius of four triangles of intersection, and is tangent to the circumcircles of the remaining four triangles of intersection.*

**5. Circles of intersection** (section 3b). a. The point $P$ is a common vertex of the two pairs of triangles $PQR$, $P'Q'R'$:$PQR'$, $PQ'R$, and the center $D$ of the

circle $(D)$ lies on the four lines $s$, $s_p$, $QR$, $Q'R'$, hence the circle $(D)$ is an Apollonian circle of each of the first two triangles and thus orthogonal to their circumcircles $(I)$, $(I_p)$ at $P$; and $(D)$ is tangent at $P$ to the circumcircles $(I'_r)$, $(I'_q)$ of the second pair of triangles (section 4). The circle $(D')$, on the other hand, is tangent to the circles $(I)$, $(I_p)$ and orthogonal to the circles $(I'_r)$, $(I'_q)$, at the point $P$. Similarly for the other points of intersection of the circles $(A)$, $(B)$, $(C)$ taken in pairs. Now the two circles of antisimilitude $(D)$, $(D')$ of the two given circles $(B)$, $(C)$ are orthogonal, hence: *The four circumcircles of four triangles of intersection having a common vertex may be grouped into two pairs such that the two circles of each group are tangent to each other, and the circles of one group are orthogonal to those of the other group.*

Observe that of the two circles of antisimilitude passing through the point of intersection considered one belongs to one of the two groups of circumcircles, and the other circle of antisimilitude belongs to the other group.

b. Consider now one of the triangles of intersection, say, $PQR$. Its circumcircle $(I)$ is tangent to three other circles of intersection, one at each of the points $P$, $Q$, $R$ (section 5a). The circle tangent to $(I)$, say, at $Q$ is the circumcircle $(I_q)$ of the triangle $P'QR'$ obtained by replacing in $PQR$ the other two vertices $P$, $R$ by their complementary points of intersection (section 2) $P'$, $R'$.

Through each vertex of $PQR$ pass two circles of intersection orthogonal to $(I)$. But the two circles orthogonal to $(I)$, say, at $P$ cut $(I)$ again in the points $Q$, $R$, respectively, hence $(I)$ is orthogonal to three other circles of intersection. The circle orthogonal to $(I)$ at the points, say, $P$, $R$ is the circumcircle $(I'_q)$ of the triangle $PQ'R$ obtained from $PQR$ by replacing the third vertex $Q$ by its complementary point of intersection $Q'$.

Summing up: *Given three mutually orthogonal circles, each of their eight circles of intersection touches three of the remaining seven circles, at the vertices of its inscribed triangle of intersection; it cuts orthogonally three other circles of intersection, at the vertices of its inscribed triangle taken two-by-two, and has no points in common with the eighth circle of intersection* (section 4b iv).

**6. Circumcenters.** a. The two triads of circles $(I)$, $(I_p)$, $(D')$; $(I'_q)$, $(I'_r)$, $(D)$ (section 5a) pass through the point $P$, and the circles of each triad are tangent to each other at that point, hence we have the two tetrads of collinear points $P$, $I$, $I_p$, $D'$; $P$, $I'_p$, $I'_r$, $D$.

Considering the point $P'$ we obtain, in a like fashion, the two tetrads of collinear points $P'$, $I'$, $I'_p$, $D$; $P'$, $I_q$, $I'$, $D'$. Thus the circumcenters of the eight triangles of intersection lie by twos on the four lines $DP$, $DP'$, $D'P$, $D'P'$.

Similarly for the other two pairs of centers of similitude $E$, $E'$; $F$, $F'$. Thus: *Given three mutually orthogonal circles, the circumcenters of their eight triangles of intersection lie by twos on the four lines joining the points of intersection of any two of the three given circles to the two centers of similitude of those two circles.*

b. COROLLARY. *If of three mutually orthogonal real circles, two circles are fixed*

*and the third circle varies, the circumcenters of their eight triangles of intersection describe four fixed straight lines.*

c. The four chords $QR$, $Q'R'$; $QR'$, $Q'R$ of the circle $(A)$ (see table B) are each the base of two triangles of intersection, namely, $PQR$, $P'QR$; $PQ'R'$, $P'Q'R'$; $PQR'$, $P'QR'$; $PQ'R$, $P'Q'R$.

Now the perpendicular from the center $A$ of the circle $(A)$ upon $QR$ passes through the circumcenters $I$, $I_p'$ of the two triangles of intersection having the chord $QR$ for base. We have thus the triad of collinear points $A$, $I$, $I_p'$.

We obtain in a like manner the triads of collinear points $A$, $I'$, $I_p$; $A$, $I_r'$, $I_q$; $A$, $I_q'$, $I_r$.

Similarly for the centers $B$, $C$ of the given circles $(B)$, $(C)$. Thus: *The eight circumcenters of the triangles of intersection of three given mutually orthogonal circles lie in pairs on three tetrads of lines passing, respectively, through the three centers of the given circles.*

Observe that we have accounted for the $(8\times 7)\div 2=28$ lines which join the eight circumcenters two-by-two (sections 6a, 6c, and 4b iv).

**7. More on the circles of antisimilitude.** We have established a number of properties of the circles of antisimilitude of three mutually orthogonal circles, in their relation to the triangles of intersection of the three given circles. These properties permit, in turn, to find some relations of the circles of antisimilitude among themselves. Here are two examples.

a. The three coaxial circles $(D)$, $(E)$, $(F)$ are the circles of Apollonius of a triangle; the same holds for the circles $(D)$, $(E')$, $(F')$ (section 4). Hence the circle $(D)$ cuts each of the circles $(E)$, $(F)$, $(E')$, $(F')$ at an angle of 120° [1; p. 266, art. 624].

Similarly for each of the other five circles of antisimilitude. Thus: *Each of the six circles of antisimilitude of three mutually orthogonal circles cuts four of the remaining five circles at an angle of* 120°, *and is orthogonal to the fifth circle.*

b. The triad of circles $(D)$, $(E')$, $(F')$ having their centers on the axis of similitude $s_p$ are the Apollonian circles of a triangle (section 4), hence the center $D$ of $(D)$ is a center of similitude of the two circles $(E')$, $(F')$ [1; p. 263, art. 612].

Considering the axes of similitude $s_q$, $s_r$ we may show in a like manner that the points $E$, $F$ are centers of similitude of the pairs of circles $(F')$, $(D')$; $(D')$, $(E')$, respectively. Hence the line $s=DEF$ is an axis of similitude of the three circles $(D')$, $(E')$, $(F')$.

Similarly for the axes of similitude $s_p$, $s_q$, $s_r$ of the three given circles. Thus: *Each axis of similitude of three mutually orthogonal circles is also an axis of similitude of those three circles of antisimilitude of the three given circles whose centers lie outside the axis considered.*

**8. Centers of similitude of the circles of intersection.** Given three mutually orthogonal real circles: a. Their orthogonal center is a center of similitude, and their orthogonal circle is a circle of antisimilitude of each of their four pairs of complementary circles of intersection.

b. The remaining four centers of similitude of the four pairs of circles of intersection coincide with the poles of the four axes of similitude of the three given circles with respect to their orthogonal circle.

c. The same four points are also the trilinear poles of the axes of similitude considered with respect to the central triangle of the three given circles.

d. The complete quadrangle formed by those four points has the central triangle of the three given circles for its diagonal triangle.

e. The six sides of the complete quadrangle just mentioned pass respectively through the six centers of similitude of the three given circles.

The proofs of these propositions are omitted, to save space.

**9. Paraphrases.** The orthogonal center $M$ of the three mutually orthogonal circles $(A)$, $(B)$, $(C)$ is the orthocenter of the triangle $ABC$, and their orthogonal circle $(M)$ is the polar circle of $ABC$ (**4**).

The points $P, P'; Q, Q'; R, R'$ (section 2) are the points of intersection of the altitudes of the triangle $ABC$ with the circles having for diameters the respective sides of $ABC$. Some of the properties obtained for the three circles $(A)$, $(B)$, $(C)$, may be restated in terms of these elements of the triangle $ABC$. The paraphrasing of those propositions is left for the entertainment of the reader.

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# AN INTRODUCTION TO ELLIPTIC GEOMETRY

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**1. The problem.** Anyone familiar with the intuitive presentations of elliptic geometry in American and British books, even the most recent, must admit that their handling of the foundations of this subject is less than fair to the student. Once these books manage to get past ("prove" would be the wrong word) basic propositions such as *all perpendiculars to a straight line meet in a point* and *the length of a straight line is a multiple of its polar distance* the sailing is comparatively smooth, but until then the waters are, indeed, rough and muddy. For, the stated assumptions on which these theorems are based are vague, the proofs involve an unduly large number of unwarranted procedures even for an intuitive treatment, and far too little care is taken in the proofs to distinguish between the single and double elliptic situations. It is commonly assumed, for example, that a straight line is *boundless*, but this term is not defined. While all straight lines are assumed to meet, there is usually no clear statement as to the number of times they are supposed to meet, nor why they cannot meet more than twice. Nor is it made clear how much of Euclidean theory, particularly that concerning angles and the congruence of triangles, is being retained, nor under what conditions.

This article attempts to remedy the situation by offering an intuitive *introduction* in which more than traditional care is taken in preparing the student's mind for the acceptance of the new ideas, in stating basic assumptions, in carrying out proofs, and in keeping the single and double elliptic geometries separate. Although this is only intended to be a beginning, and hence does not go beyond theorems like those mentioned above, the axioms we have adopted are suitable for developing the subject much further. Only two-dimensional geometry is considered.

**2. History.** Saccheri seems to have been the first to take the revolutionary step of considering, at least tentatively, the logical possibility that there might be systems of geometry other than Euclidean, one associated with his hypothesis of the acute angle, the other with his hypothesis of the obtuse angle. He ruled out the first system as untenable by reaching contradictions where none existed, and hence failed to discover hyperbolic geometry. The hypothesis of the obtuse angle also led him to a contradiction, a result of his tacit assumption that a straight line is infinite, and so this system was ruled out too. But since he had set himself the task of developing this hypothesis on the basis of Euclid's axioms and postulates, excluding the parallel postulate, and since the infinitude of a straight line is not explicitly included in this basis, it may be claimed that Saccheri had not found a genuine contradiction in his second system.

For a century after Saccheri published his *Euclides Vindicatus* (1733) the geometers who occupied themselves with the problem of Euclid's parallel

postulate took it for granted that a straight line is infinite, and hence it seems not to have occurred to them to ask, let alone try to answer, the question: Can the hypothesis of the obtuse angle lead to a consistent system of geometry if it is not assumed that a straight line is infinite? The question would not have been at all academic when it is realized that Saccheri had already deduced some interesting consequences of this hypothesis, for example, that the angle-sum of every triangle exceeds two right angles, that the locus of points equidistant from a straight line $l$ is not a straight line but a curve convex toward $l$, and that two straight lines intersect if the sum of the interior angles on the same side of a transversal is less than two right angles.

In 1854, not long after the solution of the parallel problem had culminated in the discovery of hyperbolic geometry, the above question became unavoidable, with the pronouncement by Riemann that space need not be regarded as infinite just because it seems boundless. Attention was thus directed to the consideration of possible systems of geometry in which a straight line, although endless, is nevertheless finite in length. Such a line can be visualized as a closed path which, like a circle or ellipse, does not intersect itself. Adopting this viewpoint mathematicians before long discovered, not one, but two consistent systems of geometry in which Saccheri's hypothesis of the obtuse angle holds. When it is recalled that in Euclidean and hyperbolic geometry the existence of parallel lines is established with the aid of the assumption that a straight line is infinite, it comes as no surprise that there are no parallel lines in the two new, *elliptic geometries*.

**3. Geometry on a surface.** To make Riemann's ideas and the new geometries intuitively acceptable requires only a simple broadening of our outlook. When the physical world is viewed in the Euclidean manner, a flat surface like that of a blackboard, table top, or swimming pool is regarded as a small, somewhat imperfect portion of an ideal flat surface of infinite extent. Hence, although the straight paths on such physical surfaces, *e.g.* rulings on a blackboard, are necessarily of limited extent, we imagine them to be parts of infinite straight paths, and hence to obey the propositions of Euclidean geometry. When things are viewed in the manner of hyperbolic geometry, however, the very same blackboard can be regarded, with equal correctness, as a tiny part of an enormous pseudosphere, and the very same rulings as obeying hyperbolic geometry, for within limited regions both systems are equally correct.

But these two systems do not exhaust the ways in which we can view the physical world. Our blackboard can be regarded as a small part of many other surfaces, and its rulings as subject to the various geometries of the shortest paths on these surfaces. Among such surfaces are some, the spheres being the simplest, on which the shortest paths are closed curves of great extent. Associated with these surfaces, then, are geometries, including the elliptic types, in which the straight line is endless yet finite, in accord with Riemann's ideas. Thus, in addition to the Euclidean and hyperbolic ways of rationalizing the

blackboard and its rulings (or any flat physical surface and its straight paths) we also have the Riemannian. Indeed, the latter makes more sense than the other two if the universe is of limited extent and not "roomy" enough for their straight lines.

**4. Double elliptic geometry.** Our problem of choosing axioms for this geometry is something like what would have confronted Euclid in laying the basis for 2-dimensional geometry had he possessed Riemann's ideas concerning straight lines and used a large curved surface, with closed shortest paths, as his model, rather than an infinite flat one. And like Euclid we shall proceed in a strongly intuitive manner.

The terms "point," "line," and "length" are not defined, and it is taken for granted that a line has a finite length and contains points. A "closed line," also undefined, is to be regarded as a special kind of line. (The reader will find it useful to visualize a line as a continuous curve and a closed line as a closed continuous curve.) In turn, a "straight line" is a special kind of closed line. Thus "line" and "straight line" are not synonymous.

AXIOM 1. *A straight line is a closed line (of finite length) not intersecting itself.*

In the geometry we are building there shall be no parallel straight lines. How many times, then, shall two straight lines meet? If two closed paths drawn on a piece of paper cut each other once they will cut each other at least once more. In the simplest situation they cut each other just twice. Guided by this we assume

AXIOM 2. *Each pair of straight lines meet in exactly two points.*

Among the various paths joining two points on a physical surface there is usually at least one which is shortest. Hence we assume

AXIOM 3. *Among the lines joining two points there is one (or more) whose length is least.*

A line with this property is called a *segment*, and its length is called the *distance* between the two points. If there is exactly one segment joining two points, the segment is called *unique*. (It must be repeated that "line" and "straight line" are not synonymous.)

Axiom 1 suggests that each two points $A$ and $B$ of a straight line divide it into two parts, each a line joining $A$ and $B$. Since a straight line should be a "repository" of shortest paths, we shall want at least one of these two lines joining $A$ and $B$ to be a segment. This leads to Axiom 4. The converse leads to Axiom 5.

AXIOM 4. *A straight line is divided into two lines by each two of its points, and at least one of these lines is a segment.*

AXIOM 5. *Any given segment joining two points is contained in some straight line through the points.*

A simple consequence of Axiom 4 is:

THEOREM 1. *If the two lines into which a straight line is divided by two of its points are unequal, the lesser is a segment joining the points; if equal, both are segments.*

Two points that divide a straight line into equal segments are called *antipodal points* of the line, each being *antipodal to the other* on the line.

From Axioms 3 and 5 we infer:

THEOREM 2. *There is a straight line through each two points.*

When will a unique straight line go through two points? A partial answer is given by Theorem 3, which follows from Axioms 2, 4, and 5.

THEOREM 3. *A unique straight line goes through two points if there is a unique segment joining them.*

The rest of the answer to the above question is given as follows:

AXIOM 6. *A straight line g through two points is the only straight line through them if, and only if, the points are non-antipodal on g.*

From this we deduce Theorem 4 (the converse of Theorem 3) and Theorem 5:

THEOREM 4. *There is a unique segment joining two points if there is a unique straight line through them.*

THEOREM 5. *Two points cannot be antipodal on one straight line and non-antipodal on another.*

If a point $B$ is antipodal to a point $A$ on one straight line through $A$ it is therefore antipodal to $A$ on every other straight line through $A$. Hence we may simply say *$B$ is antipodal to $A$* without specifying a particular straight line. It follows that to any given point there corresponds exactly one other which is antipodal to it.

Let $a$ and $b$ be any two straight lines. By Axioms 2 and 4 each divides the other into two lines, at least one of which is a segment. By Axiom 6 the two lines into which $a$ is divided are equal, and hence both are segments. Likewise, the two lines into which $b$ is divided are also segments. By the definition of a segment each segment on $a$ equals each one on $b$. Thus we have

THEOREM 6. *All straight lines bisect each other and have the same length.*

From Theorem 6 we see that all segments formed by the intersection of straight lines in pairs are equal. Let us take the common length of these segments as unity. The length of a straight line is then 2 units. Also the distance

between two points cannot exceed 1. For, let $A$ and $B$ be any two points. Then, by Axiom 3 and the definition of distance, there is a segment joining them whose length is the distance between them. By Axiom 5 this segment is contained in some straight line $g$ through $A$ and $B$. If $A$ and $B$ divide $g$ equally, in which case they are antipodal, this segment is one of the two thus formed and its length is 1, as agreed above. If $A$ and $B$ divide $g$ into unequal parts, the mentioned segment must be the lesser and its length less than 1. Thus we have

THEOREM 7. *The distance between two points is maximum if, and only if, the points are antipodal. This maximum value is* 1, *and the length of a straight line is* 2.

It is easy to verify the truth of the following theorem:

THEOREM 8. *There is a unique segment, and hence a unique straight line, joining two points if, and only if, the distance between them is less than* 1.

Since a small portion of a curved surface may be nearly flat, it is clear that the surface may have certain *local* properties in common with a Euclidean plane. For example, it would not be unreasonable to expect that the angular theory at a point is one such common property. By this is meant that the total angle at a point is the same for all points, namely 360°, that the vertical angles formed by two straight lines are equal, that there is a unique perpendicular at any point of a straight line, that all right angles equal 90°, that all straight angles equal 180°, and so forth. For brevity let us assume all these properties at one stroke.

AXIOM 7. *The angular theory at a point is the same as in Euclidean plane geometry.*

Similarly, since our geometry is to be as applicable to the flat physical surfaces of experience as Euclidean geometry, we should expect to find in it some duplication, at least locally, of the Euclidean theory on the congruence of triangles. To attain this local restriction we shall understand a *triangle* to mean only a figure consisting of three non-collinear points and three segments joining them in pairs. (It easily follows that each such segment is unique.) These segments are called the *sides of the triangle*, and the three angles which they determine, each less than 180°, are the *angles of the triangle*. Calling two triangles, as usual, *congruent* if their sides and angles are equal, respectively, we then adopt

AXIOM 8. *Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*

Our final assumption, which expresses another local property of many familiar surfaces, is:

AXIOM 9. *A straight line which subdivides an angle of a triangle meets the opposite side.*

We now prove two basic theorems dealing with the perpendiculars to a straight line.

Let $b$ be any straight line, and $M$ any point on $b$ (Fig. 1). Then $c$, the straight line perpendicular to $b$ at $M$ meets $b$ again in a point $N$. Let $P$ be the midpoint of one of the two segments into which $b$ divides $c$, and denote this segment by $MPN$. Let $\beta$ be one of the two segments into which $c$ divides $b$. The straight line $d$, perpendicular to $c$ at $P$, meets $b$ in two points. By Theorem 6 neither of these points coincides with $M$ or $N$, and exactly one lies on $\beta$. Call this latter point $R$.

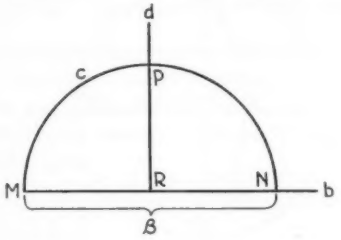


FIG. 1

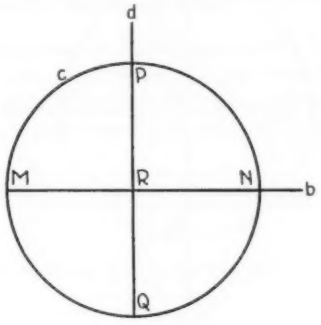


FIG. 2

If $d$ is continued beyond $R$ it will meet $c$ in a point $Q$ antipodal to $P$ by Theorem 6 (Fig. 2). Continued beyond $Q$ it will meet $b$ in a second point $S$, and continued beyond $S$ it must return to $P$ (Fig. 3). Since $c$ bisects $d$ by Theorem 6, and $b$ does also, we obtain, respectively,

$$PR + RQ = QS + SP = 1$$

and

$$RQ + QS = SP + PR = 1.$$

From this we infer that

$$PR = RQ = QS = SP = 1/2.$$

By Theorem 8 this means that $PR$, $RQ$, $QS$, and $SP$ are unique segments. Also $MR$, $NR$, $MS$, $NS$ are unique segments, each being clearly less than 1. Hence we obtain triangles $MPR$ and $NPR$, and they are congruent since $MP = NP$, $PR = PR$, $\sphericalangle MPR = \sphericalangle NPR$. Hence $\sphericalangle MRP$ is a right angle. Similarly, we obtain triangles $MPS$ and $NPS$, and they too are congruent by side-angle-side. Hence $\sphericalangle MSP$ is a right angle.

Thus $d$, like $c$, goes through $P$ and $Q$, and is perpendicular to $b$ at points each $\frac{1}{2}$ unit from $P$ and $\frac{1}{2}$ unit from $Q$. The same is true of every straight line $e$ through $P$ (Fig. 4). Thus if $e$ goes through $P$ so as to subdivide $\sphericalangle MPR$ of triangle $MPR$, it must meet side $MR$ in a point $T$ by Axiom 9, and cannot meet sides $MP$ or $RP$ by Theorem 6, since it goes through $Q$. Since $e$, when extended beyond $T$, goes through $Q$, it must subdivide one of the two pairs of vertical

angles at $Q$ formed by $c$ and $d$. If it subdivided $\sphericalangle RQN$ of triangle $RQN$ it would meet side $RN$ in a point by Axiom 9, but then $b$ and $e$ would not bisect each other, contradicting Theorem 6. Hence $e$ subdivides $\sphericalangle MQR$.

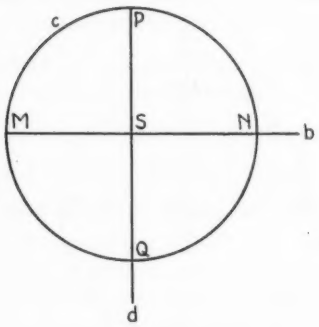


FIG. 3

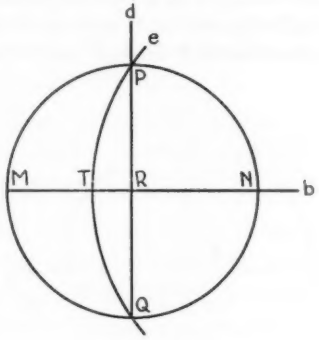


FIG. 4

We can now extend $e$ beyond $Q$, proceed just as we did for $d$, and prove that $b$ and $c$ divide $e$ into four equal parts, each a unique segment. If $PT$ and $QT$ in Figure 4 are two of the four parts then we obtain triangles $RTP$ and $RTQ$, and they are congruent since $RP=RQ$, $\sphericalangle PRT=\sphericalangle QRT$, $RT=RT$. Hence $\sphericalangle RTP=\sphericalangle RTQ=$a right angle. Likewise for the two other parts. Finally, let $X$ be any point of $b$, and $x$ the straight line perpendicular to $b$ at $X$. Since $X$ is not antipodal to $P$ there is only one straight line $PX$ according to Theorems 7 and 8, and since this straight line goes through $P$ it is perpendicular to $b$ at $X$, in accordance with the above discussion. There being only one straight line perpendicular to $b$ at $X$, straight lines $x$ and $PX$ are identical. Hence we have

THEOREM 9. *If $b$ is any straight line, each straight line $c$ perpendicular to $b$ goes through the same two points $P$ and $Q$, and is divided by these points and $b$ into four equal segments. The points in which $c$ meets $b$ are each $\frac{1}{2}$ unit from $P$ and $Q$.*

Let $P$ be any point. If $M$ is a point $\frac{1}{2}$ unit from $P$ there is a unique straight line $c$ through $M$ and $P$ by Theorem 8. Let $b$ be the straight line perpendicular to $c$ at $M$. If $X$ is any point $\frac{1}{2}$ unit from $P$, it follows from the proof of Theorem 9 that straight line $PX$ is perpendicular to $b$. Hence $X$ is on $b$ by Theorem 9, and we have

THEOREM 10. *The locus of points $\frac{1}{2}$ unit from any point $P$ is a straight line which is perpendicular to each straight line through $P$.*

It was not intended to develop the subject beyond this point.

**5. Single elliptic geometry.** The remarks made in the opening paragraph of

article 4 apply here word for word and are omitted in the interest of brevity. It is worth repeating, however, that "line" and "straight line" are not synonymous. Some of the motivations are also the same as those previously given for double elliptic geometry and will be omitted.

AXIOM 1. *A straight line is a closed line (of finite length) not intersecting itself.*

AXIOM 2. *Each pair of straight lines meet in exactly one point.*

The preceding axiom is simple in the sense that it provides us with a familiar state of affairs, *i.e.*, that prevailing in Euclidean geometry. On the other hand it shows that the surface on which single elliptic geometry holds will have to possess some unusual features in order to contain closed lines cutting each other exactly once.

AXIOM 3. *Among the lines joining two points there is one (or more) whose length is least.*

The terms *segment*, *unique segment*, and *distance* mean the same as in double elliptic geometry.

AXIOM 4. *A straight line is divided into two lines by each two of its points, and at least one of these lines is a segment.*

AXIOM 5. *Any given segment joining two points is contained in some straight line through the points.*

From Axiom 4 we infer

THEOREM 1. *If the two lines into which a straight line is divided by two of its points are unequal, the lesser is a segment joining the points; if equal, both are segments.*

It is convenient to define *antipodal points* on a straight line as in double elliptic geometry. From Axioms 2 and 5 we obtain

THEOREM 2. *Exactly one straight line goes through each two points.*

In view of this theorem, if $B$ is antipodal to $A$ on one straight line through $A$, and $C$ is antipodal to $A$ on a second straight line through $A$, then $B$ and $C$ are distinct, unlike the case in double elliptic geometry.

AXIOM 6. *The angular theory at a point is the same as in Euclidean plane geometry.*

We change the definition of a triangle given for double elliptic geometry to the extent of permitting its sides to be *any* portions of straight lines, not necessarily segments. Retaining the previous definition for congruence of triangles, we make the following assumption:

AXIOM 7. *Two triangles are congruent if two angles and the included side of one are equal, respectively, to two angles and the included side of the other.*

Two straight lines which are perpendicular to the same straight line will meet by Axiom 2. Hence there are triangles containing two right angles. Concerning such triangles we make the following reasonable assumption, our last:

AXIOM 8. *If two angles of a triangle are right angles, the opposite sides are equal.*

This axiom may call for some comment. In Euclidean geometry the theorem that the sides opposite equal angles are equal results from the fact that there is a unique perpendicular from a point to a straight line. In single elliptic geometry the latter property does not hold, as we shall see, and yet it is true that sides opposite equal angles are equal. Hence we have assumed the latter property, though in a restricted form. In double elliptic geometry, however, we were able to prove the content of Axiom 8.

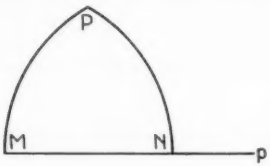


FIG. 5

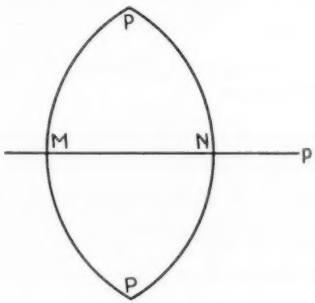


FIG. 6

Now let $p$ be any straight line, $M$ and $N$ any two of its points, and $MN$ one of the two lines into which $M$ and $N$ divide $p$. The straight lines perpendicular to $p$ at $M$ and $N$ meet in a point $P$. Since straight lines are closed lines there is not just one triangle $MPN$. Hence let us fix our attention on a triangle $MPN$ of which the chosen part $MN$ is a side (Fig. 5). If we extend sides $PM$ and $PN$ beyond $M$ and $N$, respectively, until we return to $P$, and represent things conveniently as in Figure 6, a second triangle $MPN$ is obtained having $MN$ in common with the first. These triangles are congruent by Axiom 7, so that the two sides $PM$ are equal, and likewise for the two sides $PN$. Hence $P$ is antipodal to $M$ on straight line $PM$, and to $N$ on straight line $PN$. Since $M$ and $N$ are any two points of $p$ we may think of $M$ as fixed and $N$ as variable. From what was just shown it then follows that regardless of the position of $N$ the straight line perpendicular to $p$ at $N$ must meet straight line $MP$ in the point of the latter which is antipodal to $M$, namely, in $P$. Conversely, using this fact, together with Axiom 2 and Theorem 2, we infer that each straight line through $P$ is perpendicular to $p$. We have thus proved:

THEOREM 3. *All the perpendiculars to a straight line $p$ go through the same*

*point $P$, and each straight line through $P$ is perpendicular to $p$, meeting the latter in a point which is antipodal to $P$.*

We call $P$ the *pole* of $p$, and $p$ the *polar* of $P$. In view of Theorem 3 not only does any given straight line have a pole, but any given point is the pole of some straight line. It follows that the locus of points antipodal to a given point is a straight line, the polar of the latter. In double elliptic geometry, on the other hand, this locus is a single point.

If we apply Axiom 8 to the triangle of Figure 5 we infer that sides $PM$ and $PN$ are equal. Since these sides are segments by Theorems 1 and 3, and $M$ and $N$ are any points of $p$, we see that $P$ is at a fixed distance from the points of $p$. We call this distance the *polar distance for* $p$ and denote it by $\alpha$. Using what was shown in connection with Figure 6 we infer that all straight lines perpendicular to $p$ have the same length, namely, $2\alpha$. Now let $q$ and $r$ be two straight lines through $P$ which are perpendicular to each other. By Theorem 3 they are also perpendicular to $p$ and hence have the same length $2\alpha$. Now $r$, like $p$, has the property that all straight lines perpendicular to it have a common length. Hence $p$ and $q$, being perpendicular to $r$, have the same length. Thus $p$, as well as each straight line perpendicular to it, has the length $2\alpha$. Finally, let $p'$ be any straight line distinct from $p$, let $P'$ be the pole of $p'$, and let $2\alpha'$ be the common length of $p'$ and of each straight line perpendicular to it. If we choose equal segments $MN$ and $M'N'$ on $p$ and $p'$, respectively, then triangles $MNP$ and $M'N'P'$ are congruent by Theorem 3 and Axiom 7. Hence sides $MP$ and $M'P'$ are equal, *i.e.*, $\alpha=\alpha'$. Thus $p$ and $p'$ have equal polar distances and, consequently, equal lengths.

THEOREM 4. *All straight lines have the same polar distance $\alpha$ and the same length $2\alpha$.*

If $A$ and $B$ are any two points, they are either antipodal or non-antipodal on the straight line determined by them, and hence the distance between them is either $\alpha$ or less than $\alpha$. The maximum distance between any two points is then $\alpha$. We thus have

THEOREM 5. *The distance between two points is maximum if, and only if, the points are antipodal. This maximum equals $\alpha$, the common polar distance for all straight lines.*

It was not intended to develop the subject any further.

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